Lecture 10

Thursday, October 10, 2024 11:32

Last Lecture: discussed the angular momentum operators.

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

cyclic permutation

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

$$[\hat{L}^2, \hat{L}_x] = ?$$

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_x]$$

$$= \cancelto{0}{[\hat{L}_x^2, \hat{L}_x]} + [\hat{L}_y^2, \hat{L}_x]$$

$$= \cancel{[L_x, L_x]} + [L_y, L_x]$$

$$+ [\hat{L}_z^2, L_x]$$

$$= \hat{L}_y[\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y$$

$$+ \hat{L}_z[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_z$$

$$= -\cancel{i\hbar\hat{L}_y\hat{L}_z} - \cancel{i\hbar\hat{L}_z\hat{L}_y}$$

$$+ \cancel{i\hbar\hat{L}_z\hat{L}_y} + \cancel{i\hbar\hat{L}_y\hat{L}_z}$$

$$\boxed{[\hat{L}^2, \hat{L}_x] = 0}$$

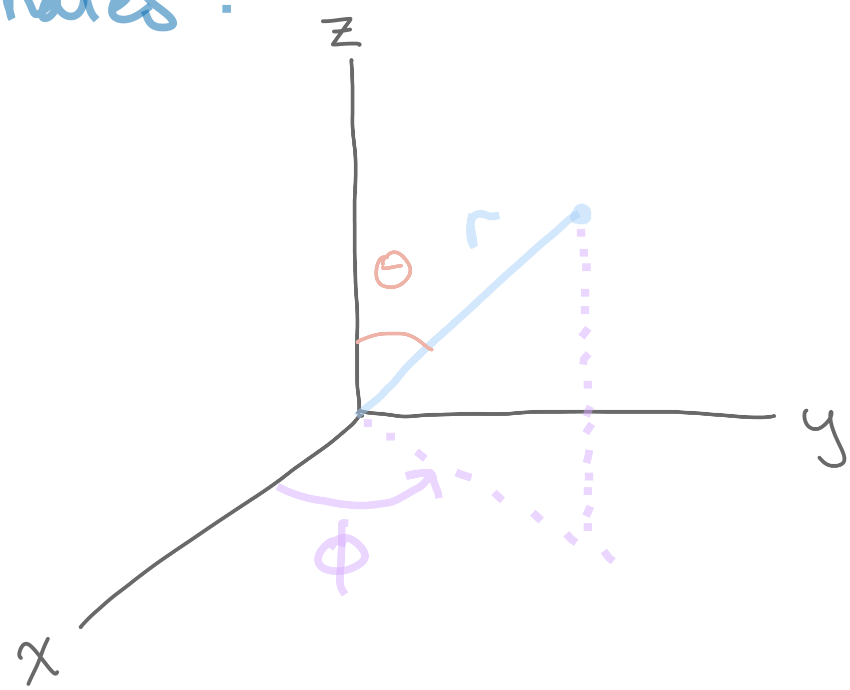
$$\boxed{[\hat{L}^2, \hat{L}_y] = 0}$$

$$\boxed{[\hat{L}^2, \hat{L}_z] = 0}$$

Thus, we can assign definite values to L^2 and any one of (L_x, L_y, L_z)

simultaneouly. But because no two components of $\hat{L}$ commute with each other, we cannot specify more than one component simultaneously.

Need to transform to spherical coordinates:



$$0 \leq r \leq \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi \leq 2\pi$$

$$x = r \sin\theta \cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r^2 = x^2 + y^2 + z^2$$

$$\cos\theta = \frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$\tan\phi = \frac{y}{x}$$

(Read pages 102–103 for details of the transformation.)

$$\hat{L}_x = -i\hbar\left[r\sin\theta\sin\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - r\cos\theta\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right]$$

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}}$$

$$\boxed{\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}$$

One-Particle Orbital Angular Momentum Eigenfunctions and Eigenvalues

$\hat{L}_z$ and $\hat{L}^2$ have a common set of eigenfunctions, $Y(\theta,\phi)$.

$$\hat{L}_z Y(\theta,\phi) = bY(\theta,\phi)$$

$$\hat{L}^2 Y(\theta,\phi) = cY(\theta,\phi)$$

$$Y(\theta,\phi) = S(\theta)T(\phi)$$

(separation of variables)

$$-i\hbar\frac{\partial}{\partial\phi}\left[S(\theta)T(\phi)\right] = b\left[S(\theta)T(\phi)\right]$$

$$-i\hbar \cancel{S(\theta)} \frac{dT(\phi)}{d\phi} = b \cancel{S(\theta)} T(\phi)$$

$$\frac{dT(\phi)}{T(\phi)} = \frac{ib}{\hbar} d\phi$$

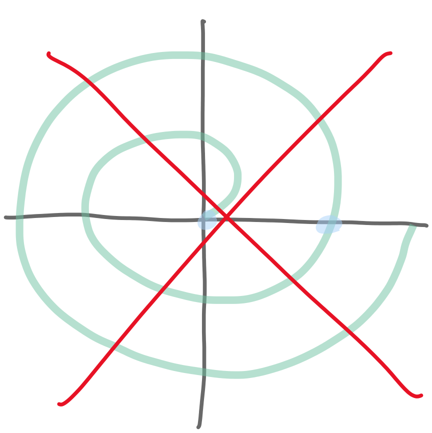
$$\int \frac{1}{T(\phi)} dT(\phi) = \frac{ib}{\hbar} \int d\phi$$

$$\ln T(\phi) = \frac{ib}{\hbar} \phi + C$$

$$\boxed{T(\phi) = A e^{ib\phi/\hbar}}$$

We need to ensure that

$$T(\phi + 2\pi) = T(\phi)$$



not valid

$$Ae^{ib\phi/\hbar}e^{ib2\pi/\hbar} = Ae^{ib\phi/\hbar}$$

$$e^{ib2\pi/\hbar} = 1$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha = 1, \quad \alpha = 2\pi m$$

$$m = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\frac{b}{\hbar} = m$$

$$b = m\hbar, \quad m = \ldots, -2, -1, 0, 1, 2, \ldots$$

$$\boxed{T(\phi) = Ae^{im\phi}, \quad m = 0, \pm 1, \pm 2, \ldots}$$

$$d\tau = r^2\, dr \sin\theta\, d\theta\, d\phi$$

$$d\tau = dx\, dy\, dz$$

$$\int_0^\infty \left[\int_0^\pi \left(\int_0^{2\pi} |F(r,\theta,\phi)|^2 d\phi\right)\sin\theta\, d\theta\right] r^2 dr = 1$$

$$\int_0^\infty |R(r)|^2 r^2 dr \int_0^\pi |S(\theta)|^2 \sin\theta \, d\theta \int_0^{2\pi} |T(\phi)|^2 d\phi$$

$$= 1$$

$$\int_0^\infty |R(r)|^2 r^2 dr = 1$$

$$\int_0^\pi |S(\theta)|^2 \sin\theta \, d\theta = 1$$

$$\boxed{\int_0^{2\pi} |T(\phi)|^2 d\phi = 1}$$

$$\int_0^{2\pi} \left(Ae^{im\phi}\right)^* Ae^{im\phi} \, d\phi = 1$$

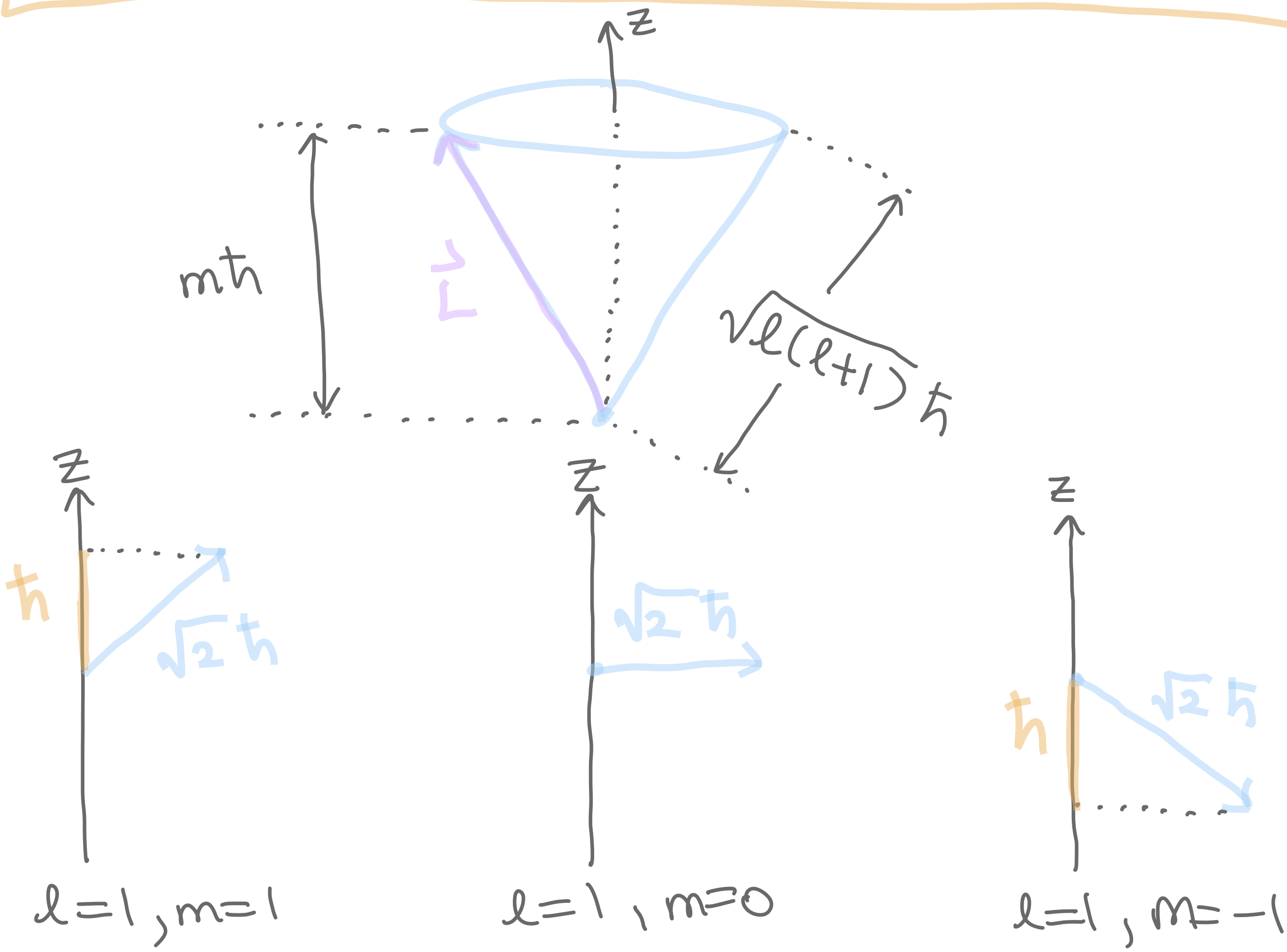
$$|A|^2 \int^{2\pi} e^{-im\phi} e^{im\phi} \, d\phi = 1$$

0

$$|A|^2 \int_0^{2\pi} d\phi = 1$$

$$|A| = \frac{1}{\sqrt{2\pi}}$$

$$\boxed{T(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \dots}$$



For $\hat{L}^2$, we have $\hat{L}^2 Y = cY$

$$-\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)S(\theta)\frac{1}{\sqrt{2\pi}}e^{i}$$

$$= c\,S(\theta)\frac{1}{\sqrt{2\pi}}e^{im\phi}$$

$$\boxed{\frac{d^2S}{d\theta^2} + \cot\theta\frac{dS}{d\theta} - \frac{m^2}{\sin^2\theta}S = -\frac{c}{\hbar^2}}$$

Need power series solution, with cutoff after $a_k w^k$.

Recursion relation:

set to 0

$$a_{j+2} = \frac{\boxed{(j+|m|)(j+|m|+1) - c/\hbar^2}}{(j+1)(j+2)}\,c$$

$$c = \hbar^2(k+|m|)(k+|m|+1),$$

$$k = 0, 1, 2, \ldots$$

$$|m| = 0, 1, 2, \ldots$$

Define quantum number ℓ as

$$\ell \equiv k + |m|$$

$$\boxed{c = \ell(\ell+1)\hbar^2, \quad \ell = 0, 1, 2, \ldots}$$

$$|\vec{L}| = \sqrt{\ell(\ell+1)}\,\hbar$$

$$Y_\ell^m(\theta, \phi) = S_{\ell,m}(\theta)\,T(\phi)$$

$$\boxed{\underbrace{Y_\ell^m(\theta, \phi)}_{\text{spherical harmonics}} = \frac{1}{\sqrt{2\pi}}\,\underbrace{S_{\ell,m}(\theta)}_{\text{associated Legendre polynomials}}\,e^{im\phi}}$$