Lecture 11 (Midterm Exam Review)

Thursday, October 17, 2024 11:35

Problem 1: A possible wave function for an electron in a region of length l is $\sin\left(\frac{3\pi x}{l}\right)$. Normalize this wave function.

Solution: $\psi = N\sin\left(\frac{3\pi x}{l}\right)$, find N.

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$\int_0^l N^2 \sin^2\left(\frac{3\pi x}{l}\right) dx = 1$$

$$N^2 \int_0^l \sin^2\left(\frac{3\pi x}{l}\right) dx = 1$$

$$N^2 \left[\frac{1}{2}x - \frac{l}{4\cdot 3\pi}\sin\frac{6\pi x}{l}\right]_0^l = 1$$

$$N^2\left[\left(\frac{l}{2} - \frac{l}{12\pi}\sin\cancel{\frac{6\pi l}{l}}^{0}\right) - \left(\cancel{\frac{0}{2}}^{0} - \frac{l}{12\pi}\sin\cancel{\frac{6\pi\cdot 0}{l}}^{0}\right)\right] = 1$$

$$N^2\left(\frac{l}{2}\right) = 1$$

$$\boxed{|N| = \left(\frac{2}{l}\right)^{\frac{1}{2}}}$$

$$\boxed{\psi = \left(\frac{2}{l}\right)^{\frac{1}{2}}\sin\frac{3\pi x}{l}}$$

Problem 2 : What is the probability of finding the electron in the range dx at $x = \frac{l}{6}$?

Solution : $P(x) = \psi^*\psi$

$$P(x)dx = |\psi|^2 dx$$

$$(\quad)\qquad 3\pi x \leftarrow \frac{l}{6}$$

$$P(x)\,dx = \left(\frac{2}{\ell}\right) \sin^2 \frac{3\pi x}{\ell}\, dx$$

$$P(x)\,dx = \left(\frac{2}{\ell}\right) \sin^2 \frac{\cancel{3}\pi \cancel{\ell}}{2\cancel{6}\cancel{\ell}}\, dx$$

$$\boxed{P(x)\,dx = \left(\frac{2}{\ell}\right) dx}$$

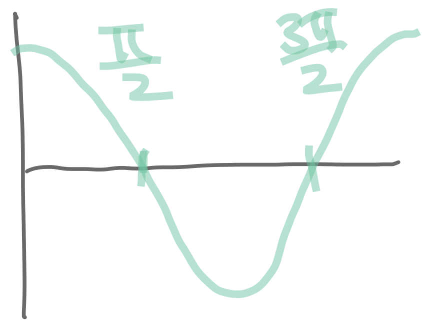
Problem 3: At what value or values of x is the probability density a maximum?

Solution: $\psi = \left(\frac{2}{\ell}\right)^{\frac{1}{2}} \sin\left(\frac{3\pi x}{\ell}\right)$

Find values of x at which $\frac{d\psi}{dx} = 0$.

$$\psi' = \left(\frac{2}{\ell}\right)^{\frac{1}{2}} \underbrace{\cos\left(\frac{3\pi x}{\ell}\right)} \cdot \frac{3\pi}{\ell}$$

When does $\cos\left(\frac{3\pi x}{\ell}\right) = 0$?



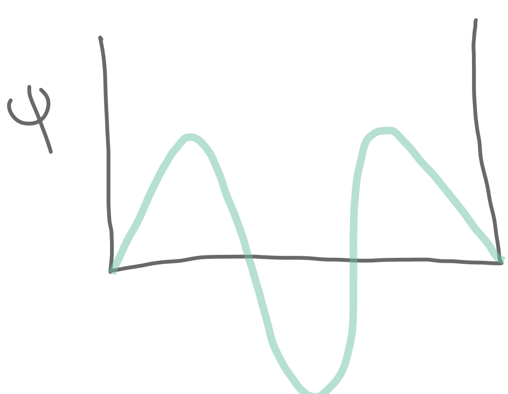
$$\frac{3\cancel{\pi}x}{\ell} = \frac{(2n+1)\cancel{\pi}}{2}$$

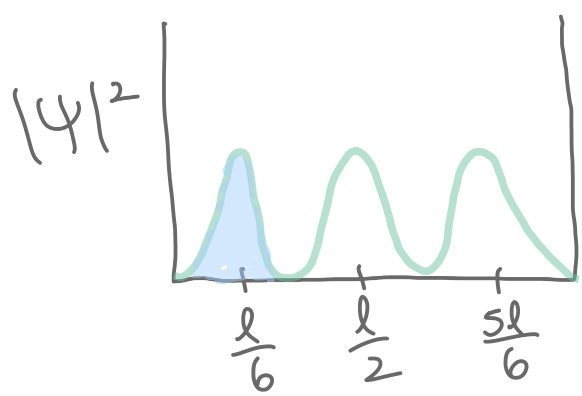
$$n = 0, 1, 2, \ldots$$

$$x = \frac{(2n+1)\ell}{6}$$

$$\boxed{\begin{aligned} n=0, \quad x &= \frac{\ell}{6} \\ n=1, \quad x &= \frac{\ell}{2} \\ n=2, \quad x &= \frac{5\ell}{6} \end{aligned}}$$

$$\cancel{n=3, \quad x = \frac{7\ell}{6}}$$ outside the box





Problem 4 : What is the probability of finding the electron between $x=0$

and $x = \frac{\ell}{3}$?

Solution: $\frac{2}{\ell}\int_0^{\ell/3} \sin^2\left(\frac{3\pi x}{\ell}\right) dx$

$$= \frac{2}{\ell}\left[\frac{1}{2}x - \frac{\ell}{12\pi}\sin\left(\frac{6\pi x}{\ell}\right)\right]_0^{\ell/3}$$

$$= \frac{2}{\ell}\left\{\left[\frac{\ell}{6} - \frac{\ell}{12\pi}\cancel{\sin\left(\frac{6\pi\ell}{3\ell}\right)}^{0}\right] - \left[\cancel{\frac{0}{2}}_{0} - \frac{\ell}{12\pi}\cancel{\sin\left(\frac{6\pi\cdot 0}{\ell}\right)}_{0}\right]\right\}$$

$$= \frac{1}{3}$$

$\boxed{\therefore \text{The probability of finding the electron between } 0 \text{ and } \frac{\ell}{3} \text{ is } \frac{1}{3}.}$

Problem 5: Identify which of the following functions are eigenfunctions of the inversion operator $\hat{\imath}$, which has the effect of making the replacement $x \to -x$:

a) $x^3 - kx$

b) $\cos kx$

c) $x^2 + 3x - 1$

Identify the eigenvalue of $\hat{\imath}$ when relevant.

Solution: a) $\hat{\imath}(x^3 - kx)$

$$= (-x)^3 - k(-x)$$

$$= -x^3 + kx$$

$$= -1(x^3 - kx)$$

This is an eigenfunction of $\hat{\imath}$ with eigenvalue -1.

b) $\hat{\imath} \cos kx = \cos k(-x)$

$$= \cos kx$$

This is an eigenfunction of $\hat{\imath}$ with eigenvalue $+1$.

c) $\hat{\imath}(x^2+3x-1)$

$= (-x)^2 + 3(-x) - 1$

$= x^2 - 3x - 1$

This is not an eigenfunction of $\hat{\imath}$.

Problem 6: A particle is in a state described by the normalized wavefunction $\psi(x) = a^{\frac{1}{2}} e^{-ax/2}$, where a is a constant and $0 \leq x \leq \infty$. Evaluate the expectation value of the commutator of the position and momentum operators.

Solution: We need to find

$$\langle[\hat{x}, \hat{p}_x]\rangle = \int^{\infty} \psi^* \hat{x}\hat{p}_x \psi \, dx$$

$$\boxed{\hat{p}_x = -i\hbar \frac{d}{dx}} \qquad - \int_0^\infty \psi^* \hat{p}_x \hat{x} \psi \, dx$$

$$\langle \hat{x}\hat{p}_x \rangle = \int_0^\infty \psi^* \hat{x}\hat{p}_x \psi \, dx$$

$$= a \int_0^\infty e^{-ax/2} x \, (-i\hbar) \frac{d}{dx} e^{-ax/2} \, dx$$

$$= -i\hbar a \int_0^\infty e^{-ax/2} x \frac{d}{dx} e^{-ax/2} \, dx$$

$$= -i\hbar a \int_0^\infty e^{-ax/2} x e^{-ax/2} \cdot -\frac{a}{2} \, dx$$

$$= \frac{i\hbar a^2}{2} \int_0^\infty x e^{-ax} \, dx$$

$$= \frac{i\hbar a^2}{2}\left[-\frac{e^{\;}(ax+1)}{a^2}\right]_0$$

$$= \frac{i\hbar a^2}{2}\left\{\cancel{\left[-\frac{e^{-a\cdot\infty}(a\cdot\infty+1)}{a^2}\right]}^{0} \right.$$

$$\left. - \left[-\frac{\underbrace{e^{-a\cdot 0}}_{1}(a\cdot 0+1)}{a^2}\right]\right\}$$

$$\langle \hat{x}\hat{p}_x\rangle = \frac{i\hbar}{2}$$

$$\langle \hat{p}_x\hat{x}\rangle = \int_0^\infty \psi^* \hat{p}_x \hat{x}\psi \, dx$$

$$= a\int_0^\infty e^{-ax/2}\, \boxed{-i\hbar}\frac{d}{dx} x e^{-ax/2} \, dx$$

$$= -i\hbar a \int_0^\infty e^{-ax/2}\left(e^{-ax/2} + x e^{-ax/2}\cdot -\frac{a}{2}\right)dx$$

$$= -i\hbar a \int_0^\infty e^{-ax/2}\left(e^{-ax/2} - \frac{ax}{2} e^{-ax/2}\right) dx$$

$$= -i\hbar a \int_0^\infty e^{-ax}\, dx - \left(-i\hbar a \frac{a}{2}\right)\int_0^\infty x e^{-ax}\, dx$$

$$= -i\hbar a \int_0^\infty e^{-ax}\, dx + \underbrace{\frac{i\hbar a^2}{2}\int_0^\infty x e^{-ax}\, dx}_{\frac{i\hbar}{2}}$$

$$= -i\hbar a \left[\frac{e^{-ax}}{-a}\right]_0^\infty + \frac{i\hbar}{2}$$

$$= -i\hbar a\left(\cancel{\frac{e^{-a\cdot\infty}}{-a}}^{\,0} - \overbrace{\frac{e^{-a\cdot 0}}{-a}}^{1}\right) + \frac{i\hbar}{2}$$

$$= -\frac{i\hbar}{2}$$

$$\langle \hat{p}_x \hat{x}\rangle = -\frac{i\hbar}{2}$$

$$\langle[\hat{x},\hat{p}_x]\rangle = \langle\hat{x}\hat{p}_x\rangle - \langle p_x x\rangle$$

$$= \frac{i\hbar}{2} - \left(-\frac{i\hbar}{2}\right)$$

$$\boxed{\langle[\hat{x},\hat{p}_x]\rangle = i\hbar}$$

$$-\frac{\hbar}{i}\cdot\frac{i}{i} = -\frac{i\hbar}{i^2} = i\hbar$$

Problem 7: Draw the vector diagram for all the permitted states of a particle with $l = 6$.

Solution: $l = 6$,

$$m = -6, -5, \dots, 0, \dots, 5, 6$$

$$|\vec{L}|^2 = l(l+1)\hbar^2, \quad |\vec{L}| = [l(l+1)]^{\frac{1}{2}}\hbar$$

$$L_z = m\hbar$$

$$|\vec{L}| = \sqrt{42}\,\hbar$$

$$L_z = -6\hbar, -5\hbar, \dots, 0, \dots, 5\hbar, 6\hbar$$