Lecture 17

Thursday, November 14, 2024 11:36

# Chapter 10 : Electron Spin and the Spin-Statistics Theorem

## 10.1 Electron Spin

Picture the electron as a sphere of charge spinning about one of its diameters.

note: electron spin has no classical analogue

Electrons have intrinsic angular momentum.

Spin angular momentum or spin.

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$$

spin angular momentum operators

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z \rightarrow [\hat{S}_x, \hat{S}_y] = i\hbar S_z$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \rightarrow [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y \rightarrow [\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

$$[\hat{S}^2, \hat{S}_x] = 0, \quad [\hat{S}^2, \hat{S}_y] = 0, \quad [\hat{S}^2, \hat{S}_z] = 0$$

Eigenvalues of $\hat{S}^2$ :

$$\boxed{s(s+1)\hbar^2, \quad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots}$$

Eigenvalues of $\hat{S}_z$ :

$$\boxed{m_s \hbar, \quad m_s = -s, -s+1, \ldots, s-1, s}$$

Quantum number $s$ : spin.

Electrons can only have $s = \frac{1}{2}$

Protons, neutrons : $s = \frac{1}{2}$

Photons : $s = 1$

(exception : photons can have $m_s = -1$ or $m_s = 1$, but not $m_s = 0$)

$s = \frac{1}{2}$

$$\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\hbar^2\right]^{\frac{1}{2}} = \frac{1}{2}\sqrt{3}\,\hbar$$

for $\hat{S}_z$, $m_s = +\frac{1}{2}\hbar$ and $m_s = -\frac{1}{2}\hbar$

$$\boxed{\hat{S}_z \alpha = +\frac{1}{2}\hbar\alpha}$$

$$\boxed{\hat{S}_z \beta = -\frac{1}{2}\hbar\beta}$$

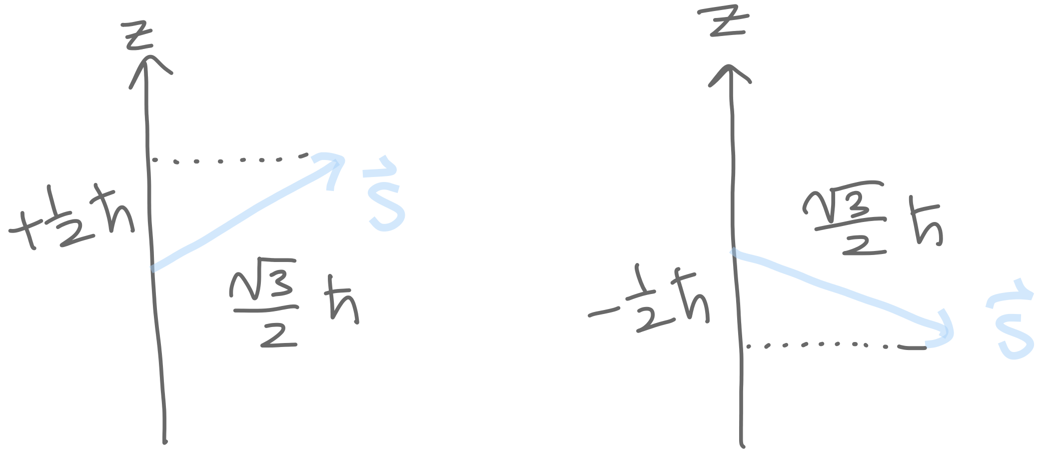
$\alpha, \beta$ are eigenfunctions of $\hat{S}_z$

Because $\hat{S}_z$ commutes with $\hat{S}^2$,

$\alpha$ and $\beta$ are also eigen functions of $\hat{S}^2$.

$$\hat{S}^2\alpha = \frac{3}{4}\hbar^2\alpha$$

$$\hat{S}^2\beta = \frac{3}{4}\hbar^2\beta$$




spin up spin down

$$\psi(x,y,z)\alpha(m_s)$$

$$\psi(x,y,z)\beta(m_s)$$

$$\sum_{m_s=-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x,y,z,m_s)|^2 dx\,dy\,dz = 1$$

# 10.2 Spin and the Hydrogen Atom

$\psi(x, y, z)\, g(m_s)$

where $g(m_s)$ is either $\alpha$ or $\beta$.

more generally,

$g(m_s) = c_1 \alpha + c_2 \beta$

$\hat{H}\left[\psi(x, y, z)\, g(m_s)\right]$

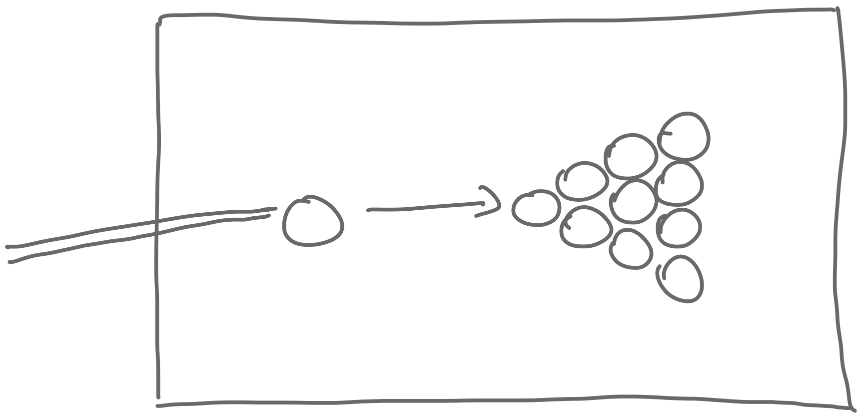
$= g(m_s)\, \hat{H}\, \psi(x, y, z)$

$= E\left[\psi(x, y, z)\, g(m_s)\right]$

$\psi(x, y, z)\alpha$ or $\psi(x, y, z)\beta$

Degeneracy of H-atom energy levels becomes $2n^2$ rather than $n^2$.

# 10.3 The Spin-Statistics Theorem



classical picture

Even if billiard balls were identical, we can still trace their individual paths on a macroscopic level.

For microscopic particles, it is not possible to trace their paths because of the uncertainty principle.

indistinguishability of electrons in the wavefunction

Consider a system of n particles:

particle 1 : $x_1, y_1, z_1, m_{s1}$

Let $q_1$ denote all four variables

(q is typically used to denote generalized coordinates)

Exange or permutation operator $\hat{P}_{12}$:

interchanges all the coordinates of particles 1 and 2.

$$\boxed{\hat{P}_{12}\,\psi(q_1, q_2, q_3, \ldots, q_n) = \psi(q_2, q_1, q_3, \ldots, q_n)}$$

Example :

$$\hat{P}_{12}[1s(1)\alpha(1)3s(2)\beta(2)] = 1s(2)\alpha(2)3s($$

What are the eigenvalues of $\hat{P}_{12}$?

$$\hat{P}_{12}\hat{P}_{12} f(q_1, q_2, \ldots, q_n)$$

$$= \hat{P}_{12} f(q_2, q_1, \ldots, q_n)$$

$$= f(q_1, q_2, \ldots, q_n)$$

applying $\hat{P}_{12}$ twice has no net effect

$$\hat{P}_{12}^2 = \hat{I}$$

Let $c_i$ and $w_i$ denote the eigenvalues and eigenfunctions of $\hat{P}_{12}$.

$$\hat{P}_{12} w_i = c_i w_i$$

$$\hat{P}_{12}^2 w_i = c_i \hat{P}_{12} w_i$$

$$\hat{I} w_i = c_i^2 w_i$$

$$c_i = \pm 1$$

$$\hat{P}_{12} w_+(q_1, q_2, \ldots, q_n) = (+1) w_+(q_1, q_2, \ldots, q$$

symmetric

$$\boxed{\hat{P}_{12}\omega_-(q_1, q_2, \ldots, q_n) = (-1)\omega_-(q_1, q_2, \ldots,}$$

antisymmetric

$$\hat{P}_{ik} f(q_1, \ldots, q_i, \ldots, q_k, \ldots, q_n)$$

$$= f(q_1, \ldots, q_k, \ldots, q_i, \ldots, q_n)$$

$$\hat{P}_{ik}\Psi(q_1, \ldots, q_i, \ldots, q_k, \ldots, q_n)$$

$$= c\Psi(q_1, \ldots, q_i, \ldots, q_k, \ldots, q_n)$$

because the particles are indisting

$\Psi$ is an eigenfunction of $\hat{P}_{ik}$

$\therefore$ The wavefunction for a system of n particles must be either symmetric or antisymmetric wit respect to interchange of any tw of the identical particles.

For electrons, the antisymmetric rule applies.

More generally, particles with half-integer spins ($s = \frac{1}{2}, \frac{3}{2}, \ldots$) require antisymmetric wavefunctio

fermions

Particles with integer spins ($s =$

require symmetric wavefunctions.

bosons

$\boxed{\text{Spin-statistics theorem}}$

$$\Psi(q_1, q_2, \ldots, q_n) = -\Psi(q_2, q_1, \ldots, q_n)$$

What if electrons 1 and 2 have the same coordinates?

i.e. $x_1 = x_2$

$y_1 = y_2$

$z_1 = z_2$

$ms_1 = ms_2$

$q_1 = q_2$

$$\Psi(q_1, q_1, \ldots, q_n) = -\Psi(q_1, q_1, \ldots, q_n)$$

$$2\Psi = 0$$

$$\Psi(q_1, q_1, \ldots, q_n) = 0$$

Two electrons with the same sp
have zero probability of being fo
at the same point in space.

$\underline{\uparrow\downarrow}$ $\underline{\uparrow\downarrow}$ $\cancel{\underline{\uparrow\uparrow}}$

Pauli exclusion

For bosons, no such restriction e

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## 10.4 The Helium Atom

For 2 electrons, we have

$\alpha(1)\alpha(2)$

$\beta(1)\beta(2)$

$\alpha(1)\beta(2)$ ] violate indistinguish

$\alpha(2)\beta(1)$

symmetric

$\alpha(1)\alpha(2)$

$\beta(1)\beta(2)$

$[\alpha(1)\beta(2)+\beta(1)\alpha(2)]\,2^{-}$

$\boxed{\text{antisymmetric } [\alpha(1)\beta(2)-\beta(1)\alpha(2)]\,2^{-}}$