Lecture 7

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Harmonic Oscillator Continued

$$y''(x) + c^2 y(x) = 0$$

using our previous method of solving the auxiliary equation, we obtain

$$y = A\cos cx + B\sin cx$$

Alternate form

$$y = D\sin(cx + e)$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$y = D\sin cx \underbrace{\cos e}_{\text{constant}} + D\cos cx \underbrace{\sin e}_{\text{constant}}$$

$$y = B\sin cx + A\cos cx$$

Series method for solving differential equations:

$$\boxed{y(x) = \sum_{n=0}^{\infty} a_n x^n}$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\boxed{y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}$$

$$2(2-1) a_2 x^0$$

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + c^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\left[(n+2)(n+1) a_{n+2} + c^2 a_n\right]}_{\text{this needs to be } 0} x^n = 0$$

$$(n+2)(n+1)a_{n+2} + c^2 a_n = 0$$

$$\boxed{a_{n+2} = -\frac{c^2}{(n+2)(n+1)} a_n}$$

recursion relation

Set $a_0 = A$, $a_1 = cB$

$$a_2 = -\frac{c^2}{(2)(1)} A, \quad a_4 = \frac{c^4}{(4)(3)(2)(1)} A, \ldots$$

$$a_3 = -\frac{c^3}{(3)(2)} B, \quad a_5 = \frac{c^5}{(5)(4)(3)(2)} B, \ldots$$

even: $a_{2k} = (-1)^k \frac{c^{2k} A}{(2k)!}, \quad k = 0, 1, 2, 3, \ldots$

odd: $a_{2k+1} = (-1)^k \frac{c^{2k+1} B}{(2k+1)!}, \quad k = 0, 1, 2, 3, \ldots$

There fore,

$$y = A \sum_{k=0}^{\infty} (-1)^k \frac{c^{2k} x^{2k}}{(2k)!} + B \sum_{k=0}^{\infty} (-1)^k \frac{c^{2k+1} x^{2k+1}}{(2k+1)!}$$

Taylor series for $\cos cx$

Taylor series for $\sin cx$

$$y = A\cos cx + B\sin cx$$

4.2 The One-Dimensional Harmonic Oscillator

Classical Model:

$$F = -kx$$

↑ force constant

$F = ma$, Newton's Second Law

$$-kx = ma \leftarrow \frac{d^2x}{dt^2}$$

$$-kx = m\frac{d^2x}{dt^2}$$

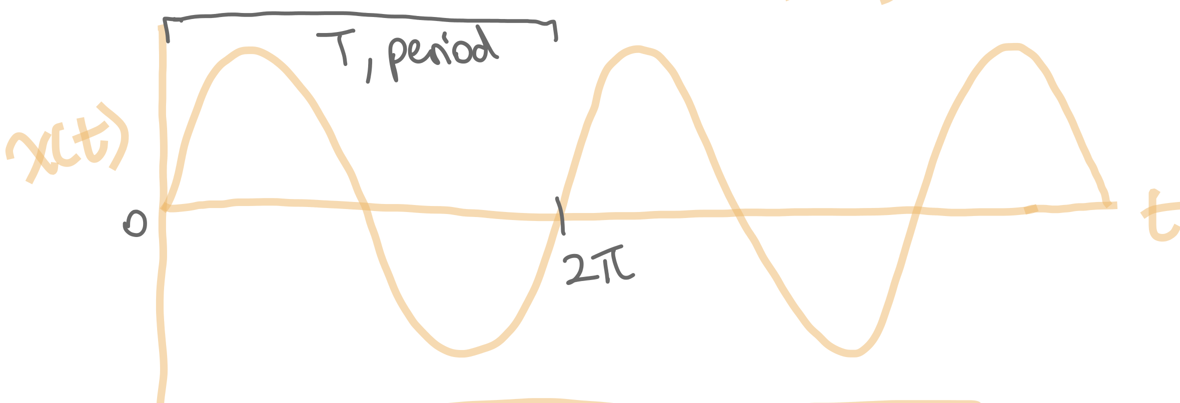
$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \qquad c^2 = \frac{k}{m}$$

$$x(t) = D\sin(ct + e)$$

$$x(t) = D\sin\left[(k/m)^{1/2}t + e\right]$$

Define $\nu = \frac{1}{2\pi}\left(\frac{k}{m}\right)^{\frac{1}{2}}$, frequency, $\frac{1}{T}$



$$\boxed{x(t) = A\sin(2\pi\nu t + b)}$$

$$x\left(t+\frac{1}{\nu}\right) = A\sin\left[2\pi\nu\left(t+\frac{1}{\nu}\right)+b\right]$$

$$x\left(t+\frac{1}{\nu}\right) = A\sin\left(2\pi\nu t + \underline{2\pi} + b\right)$$

Period $= \frac{1}{\nu}$, the time it takes for the argument of the sine function to increase by 2π.

Potential energy :

$$\boxed{F_x = -\frac{\partial V}{\partial x}}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}$$

$$F = \cancel{-}\frac{dV}{dx} = \cancel{-}kx$$

$$\int dV = \int kx\,dx$$

$$V = \frac{1}{2}kx^2 + C$$

set to 0

$$\boxed{V = \frac{1}{2}kx^2}$$

$$\nu = \frac{1}{2\pi}\left(\frac{k}{m}\right)^{\frac{1}{2}}$$

$$(2\pi\nu)^2 = \frac{k}{m}, \quad k = \boxed{4\pi^2\nu^2 m}$$

$$\boxed{V = 2\pi^2\nu^2 m x^2}$$

Kinetic energy: $T = \frac{1}{2} m v^2$

$$T = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

$$x(t) = A\sin(2\pi\nu t + b)$$

$$x'(t) = \underbrace{2\pi\nu A\cos(2\pi\nu t + b)}$$

$$T = 2\pi^2\nu^2 A^2 m\cos^2(2\pi\nu t + b)$$

$$E = T + V$$

$$= 2\pi^2\nu^2 A^2 m\cos^2(2\pi\nu t + b)$$

$$+ 2\pi^2\nu^2 m A^2\sin^2(2\pi\nu t + b)$$

$$= 2\pi^2\nu^2 m A^2 \left[\cos^2(2\pi\nu t + b)\right.$$

$$+ \sin^2(2\pi\nu t + b)]$$

$$= 1$$

$$\boxed{E = 2\pi^2\nu^2 m A^2}$$

Quantum Mechanical Treatment

$$\hat{H} = \hat{T} + \hat{V}$$

$$= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + 2\pi^2\nu^2 m x^2$$

$$\alpha \equiv \frac{2\pi\nu m}{\hbar}, \quad \alpha^2 = \frac{4\pi^2\nu^2 m^2}{\hbar^2}$$

$$= -\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} - \alpha^2 x^2\right)$$

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} - \alpha^2 x^2\right)\psi = E\psi$$

$$\left(\frac{d^2}{dx^2} - \alpha^2 x^2\right)\psi + \frac{2mE}{\hbar^2}\psi = 0$$

$$\boxed{\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \alpha^2 x^2\right)\psi = 0}$$

Make substitution to obtain a two-term recursion relation:

$$f(x) \equiv e^{\alpha x^2/2}\psi(x)$$

$$\psi(x) = \frac{f(x)}{e^{\alpha x^2/2}} = e^{-\alpha x^2/2} f(x)$$

Need to find $\psi''(x)$

$$\psi'(x) = f'(x)e^{-\alpha x^2/2} + \left(\frac{-\alpha}{2}\right)\cdot 2x\, e^{-\alpha x^2/2} f(x)$$

$$\boxed{\psi'(x) = f'(x)e^{-\alpha x^2/2} - \alpha x f(x) e^{-\alpha x^2/2}}$$

$$\psi''(x) = f''(x)e^{-\alpha x^2/2} + \underline{f'(x)e^{-\alpha x^2/2}\left(-\frac{\alpha}{2}\right)\cdot 2x}$$

$$- \Big[\alpha f(x) e^{-\alpha x^2/2} + \underline{\alpha x f'(x) e^{-\alpha x^2/2}}$$

$$+ \alpha x f(x) e^{-\alpha x^2/2} \cdot -\frac{\alpha}{2} \cdot 2x\Big]$$

$$\psi''(x) = f''(x)\underbrace{e^{-\alpha x^2/2}} - 2f'(x)\alpha x\underbrace{e^{-\alpha x^2/2}}$$
$$- \alpha f(x)\underbrace{e^{-\alpha x^2/2}} + \alpha^2 x^2 f(x)\underbrace{e^{-\alpha x^2/2}}$$

$$\boxed{\psi''(x) = e^{-\alpha x^2/2}(f'' - 2\alpha x f' - \alpha f + \alpha^2 x^2 f)}$$

$$\cancel{e^{-\alpha x^2/2}}(f'' - 2\alpha x f' - \alpha f + \cancel{\alpha^2 x^2} f)$$

$$+ (2mE\hbar^{-2} - \cancel{\alpha^2 x^2})\cancel{e^{-\alpha x^2/2}} f = 0$$

$$\boxed{f'' - 2\alpha x f' + (2mE\hbar^{-2} - \alpha) f = 0}$$

Now, try the series method:

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n x^n}$$

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \boxed{= \sum_{n=0}^{\infty} n c_n x^{n-1}}$$

$$(1)\cdot c_1 x^0 + (2) c_2 x^1 + \cdots \qquad (0) c_0 x^{-1} + (1) c_1 x^0 + (2) c_2 x^1$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2}$$

$$\boxed{f''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^{n}}$$

$$\underbrace{\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^{n}}_{f''(x)} - 2\alpha \overbrace{x \underbrace{\sum_{n=0}^{\infty} n c_n x^{n-1}}_{f'(x)}}^{x^n}$$

$$+ (2mE\hbar^{-2} - \alpha) \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{f(x)} = 0$$

$$\sum_{n=0}^{\infty} \Big[\underbrace{(n+2)(n+1)c_{n+2} - 2\alpha n c_n + (2mE\hbar^{-2} - \alpha) c_n}\Big] x^n = 0$$

$$(n+2)(n+1)c_{n+2} - 2\alpha n c_n + 2mE\hbar^{-2} c_n - \alpha c_n = 0$$

$$c_{n+2} = \frac{\alpha c_n + 2\alpha n c_n - 2mE\hbar^{-2} c_n}{(n+2)(n+1)}$$

$$\boxed{c_{n+2} = \frac{\alpha + 2an - 2mEn}{(n+2)(n+1)} c_n}$$