Lecture 8

Thursday, October 3, 2024 11:31

Harmonic Oscillator Continued

Last time:

$$C_{n+2} = \frac{\alpha + 2\alpha n - 2mE\hbar^{-2}}{(n+1)(n+2)} C_n$$

Set $C_0 = 0$:

$$\psi_1 = e^{-\alpha x^2/2} \sum_{n \text{ odd}}^{\infty} C_n x^n = e^{-\alpha x^2/2} \sum_{l=0}^{\infty} C_{2l+1} x^{2l+1}$$

Set $C_1 = 0$

$$\psi_2 = e^{-\alpha x^2/2} \sum_{n \text{ even}}^{\infty} C_n x^n = e^{-\alpha x^2/2} \sum_{l=0}^{\infty} C_{2l} x^{2l}$$

General Solution:

$$\boxed{\psi = A e^{-\alpha x^2/2} \sum_{l=0}^{\infty} C_{2l+1} x^{2l+1} + B e^{-\alpha x^2/2} \sum_{l=0}^{\infty} C_{2l} x^{2l}}$$

Boundary Conditions

Examine the ratio of successive coefficients in each series.

$$C_{2\ell+2} = \frac{\alpha + 2\alpha(2\ell) - 2mE\hbar^{-2}}{(2\ell+1)(2\ell+2)} C_{2\ell}$$

$$\frac{C_{2\ell+2}}{C_{2\ell}} = \frac{\cancel{\alpha} + 4\alpha\ell - \cancel{2mE\hbar^{-2}}}{(2\ell\cancel{+1})(2\ell\cancel{+2})}$$ if ℓ is large

$$\boxed{\frac{C_{2\ell+2}}{C_{2\ell}} \approx \frac{4\alpha\ell}{4\ell^2} = \frac{\alpha}{\ell}}$$ for ℓ large

$$C_{2\ell+3} = \frac{\alpha + 2\alpha(2\ell+1) - 2mE\hbar^{-2}}{(2\ell+2)(2\ell+3)} C_{2\ell+1}$$

$$\frac{C_{2\ell+3}}{C_{2\ell+1}} = \frac{\cancel{\alpha} + 4\alpha\ell + \cancel{2\alpha} - \cancel{2mE\hbar^{-2}}}{(2\ell\cancel{+2})(2\ell\cancel{+3})}$$ if ℓ large

$$\boxed{\frac{C_{2\ell+3}}{C_{2\ell+1}} \approx \frac{\alpha}{\ell}}$$ for large ℓ

Consider the series expansion for e^z

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

$$e^{\alpha x^2} = 1 + \alpha x^2 + \frac{\alpha^2 x^4}{2!} + \cdots$$

$$+ \boxed{\frac{\alpha^{\ell} x^{2\ell}}{\ell!}} + \boxed{\frac{\alpha^{\ell+1} x^{2\ell+2}}{(\ell+1)!}} + \cdots$$

$$\frac{\alpha^{\cancel{\ell+1}}}{(\ell+1)!} \cdot \frac{\ell!}{\cancel{\alpha^{\ell}}} = \frac{\alpha}{\ell+1} \approx \boxed{\frac{\alpha}{\ell}}$$

$$\frac{\cancel{\ell} \cdot \cancel{(\ell-1)}\cancel{(\ell-2)}\cdots}{(\ell+1)\cancel{(\ell)}\cancel{(\ell-1)}\cdots}$$

For large x, each series behaves as $e^{\alpha x^2}$.

$$\psi \approx A e^{-\alpha x^2/2} e^{\alpha x^2} + B e^{-\alpha x^2/2} e^{\alpha x^2}$$

$$e^{\alpha x^2 - \alpha x^2/2} = e^{\alpha x^2/2}$$

$$\boxed{\psi \approx A e^{\alpha x^2/2} + B e^{\alpha x^2/2}} \text{ for large } x$$

To ensure the wavefunction is quadratically integrable, break off quadratically integrable, cut off

the series after a finite number of terms.

$$\lim_{x\to\infty} x^p e^{-\alpha x^2/2} = ?$$

$$\lim_{x\to\infty} \frac{x^p}{e^{\alpha x^2/2}}$$

apply l'Hôpital's rule

$$\lim_{x\to\infty} \frac{p x^{p-1}}{\alpha x \, e^{\alpha x^2/2}}$$

apply l'Hôpital's rule

$$\lim_{x\to\infty} \frac{p(p-1) x^{p-2}}{\alpha^2 x^2 e^{\alpha x^2/2}}$$

apply l'Hôpital's rule p−2 more.

$$\lim_{x\to\infty} \frac{p!}{\alpha^p x^p e^{\alpha x^2/2}} = 0$$

How do we determine this value of n? Set $n=v$, and all $C_n > C_v$ to 0.

$$C_{n+2} = \frac{\alpha + 2\alpha n - 2mE\hbar^{-2}}{\underbrace{(n+1)(n+2)}_{\text{set to 0}}} C_n$$

$$\alpha + 2\alpha n - 2mE\hbar^{-2} = 0$$

$$\alpha + 2\alpha v - 2mE\hbar^{-2} = 0$$

$$E = \frac{\alpha + 2\alpha v}{2m\hbar^{-2}}$$

$$\alpha \equiv \frac{2\pi\nu m}{\hbar}$$

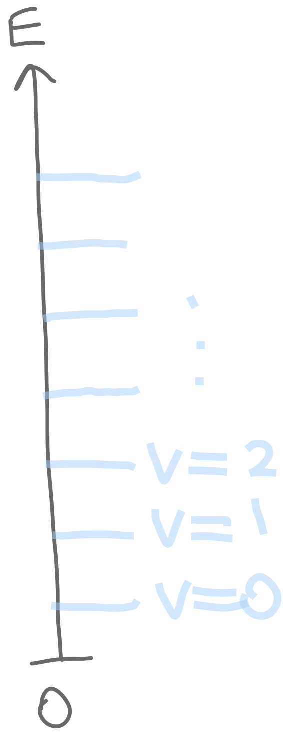
$$E = \frac{\alpha(1+2v)\hbar^2}{2m}$$

$$= \frac{\cancel{2}\pi\nu \cancel{m} \cancel{\hbar^{-1}}(1+2v)\hbar^{\cancel{2}}}{}$$

$$\cancel{2m}$$

$$= 2\pi\nu\hbar\left(v+\frac{1}{2}\right) \qquad \hbar = \frac{h}{2\pi}$$

$$\boxed{E = \left(v+\frac{1}{2}\right)h\nu\,, \quad v = 0, 1, 2, \ldots}$$



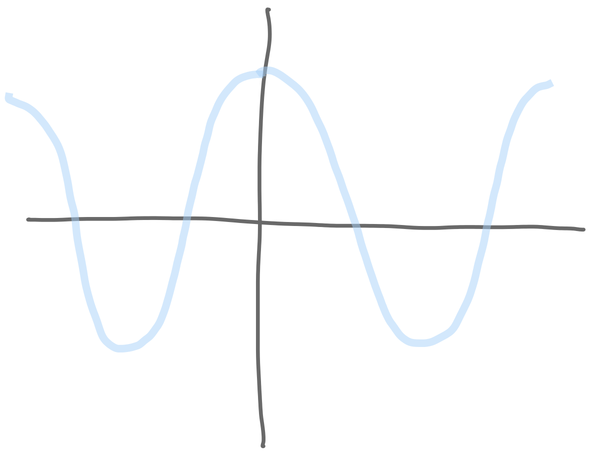
← zero-point energy in the ground state

$$\boxed{\psi_v = \begin{cases} e^{-\alpha x^2/2}\left(c_0 + c_2x^2 + \cdots + c_vx^v\right), & v \text{ even} \\ e^{-\alpha x^2/2}\left(c_1x + c_3x^3 + \cdots + c_vx^v\right), & v \text{ odd} \end{cases}}$$

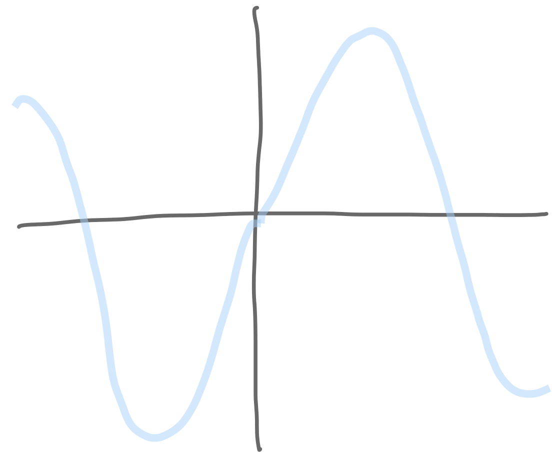
Even and Odd Functions

$f(-x) = f(x)$ even function

$f(-x) = -f(x)$ odd function



$\cos(x)$



$\sin(x)$

$$\int_{-a}^{+a} f(x)\,dx = 2\int_{0}^{+a} f(x)\,dx$$

if $f(x)$ even

$$\int_{-a}^{a} f(x)\,dx = $$

if $f(x)$ odd

Suppose we have $f(x)$ odd and $g(x)$ even.

$$f(-x)f(-x) = -f(x)-f(x) = f(x)f($$

$$f(-x)g(-x) = -f(x)\,g(x)$$

$$g(-x)g(-x) = g(x)\,g(x)$$

$$y(-x)y(-x) = y(x)y(x)$$

Let's find the explicit forms of th
wavefunctions for the three lowest lev
For $v=0$, we have

$$\psi_0 = c_0 e^{-\alpha x^2/2}$$

$$1 = \int_{-\infty}^{\infty} |c_0|^2 e^{-\alpha x^2} dx$$

$$= 2|c_0|^2 \int_0^{\infty} e^{-\alpha x^2} dx$$

use integral table :

$$\int_0^{\infty} e^{-bx^2} dx = \frac{1}{2}\left(\frac{\pi}{b}\right)^{\frac{1}{2}}, \quad b>0$$

$$1 = \cancel{2} \cdot |c_0|^2 \cdot \frac{1}{\cancel{2}}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}$$

$$|c_0| = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}$$

$$\boxed{\psi_0 = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha x^2/2}}$$

$$v = 1, \quad \psi_1 = c_1 x e^{-\alpha x^2/2}$$

$$1 = 2|c_1|^2 \int_0^\infty x^2 e^{-\alpha x^2} dx$$

Use integral table :

$$\int_0^\infty x^{2n} e^{-bx^2} dx = \frac{(2n)!}{2^{2n+1} n!}\left(\frac{\pi}{b^{2n+1}}\right)^{\frac{1}{2}}, \quad b>0$$

$n = 1, 2, 3,$

$n=1$

$$1 = 2|c_1|^2 \frac{2!}{2^3 \cdot 1!}\left(\frac{\pi}{\alpha^3}\right)^{\frac{1}{2}}$$

$$1 = |c_1|^2 \frac{4}{8}\left(\frac{\pi}{\alpha^3}\right)^{\frac{1}{2}}$$

$$|c_1| = \left(\frac{4\alpha^3}{\dots}\right)^{\frac{1}{4}}$$

$|c_1| = \left(\quad \pi \quad \right)$

$$\boxed{\psi_1 = \left(\frac{4\alpha^3}{\pi}\right)^{\frac{1}{4}} x e^{-\alpha x^2/2}}$$

$v = 2$

$$\psi_2 = (c_0 + c_2 x^2) e^{-\alpha x^2/2}$$

from recursion relation,

$$c_{n+2} = \frac{2\alpha(n-v)}{(n+1)(n+2)} c_n \qquad n=0, \quad v=2$$

$$c_2 = \frac{2\alpha(-2)}{(1)(2)} c_0 = -2\alpha c_0$$

$$\psi_2 = [c_0 + (-2\alpha c_0) x^2] e^{-\alpha x^2/2}$$

$$\psi_2 = c_0 (1 - 2\alpha x^2) e^{-\alpha x^2/2}$$

$$1 = 2 \cdot |c_0|^2 \int_0^\infty (1 - 2\alpha x^2)^2 e^{-\alpha x^2} dx$$

$$(1-2\alpha x^2)(1-2\alpha x^2)$$

$$= 1 - 4\alpha x^2 + 4\alpha^2 x^4$$

$$1 = 2\cdot|C_0|^2\left(\int_0^\infty e^{-\alpha x^2}\,dx - \int_0^\infty 4\alpha x^2 e^{-\alpha x^2}\,d\right.$$

$$\left. + \int_0^\infty 4\alpha^2 x^4 e^{-\alpha x^2}\,dx\right)$$

$$1 = 2\cdot|C_0|^2\left[\frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} - 4\alpha\cdot\frac{2}{8}\left(\frac{\pi}{\alpha^3}\right)^{\frac{1}{2}}\right.$$

$$(\alpha^2)^{\frac{1}{2}}$$

$$\left. + 4\alpha^2\frac{4!}{2^5\cdot 2!}\left(\frac{\pi}{\alpha^5}\right)^{\frac{1}{2}}\right]$$

$$\frac{24}{32\cdot \cancel{2}}\cdot \overset{2}{\cancel{4}} = \frac{48}{32} =$$

$$(\alpha^4)^{\frac{1}{2}}$$

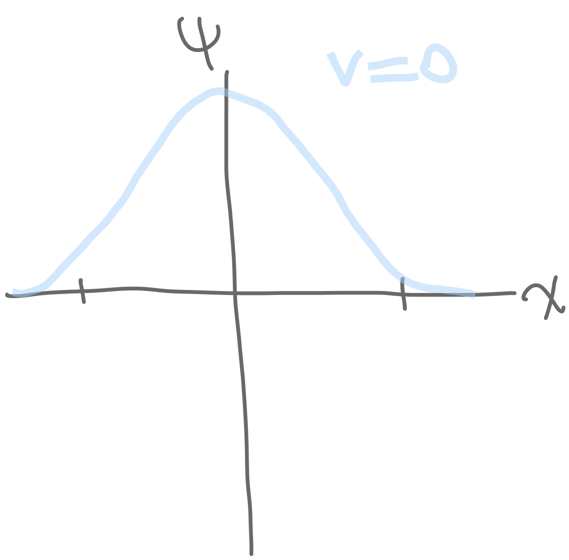
$$1 = 2\cdot|C_0|^2\left[\frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} - \left(\frac{\pi\cancel{\alpha^2}}{\alpha^{\cancel{3}}}\right) + \frac{3}{2}\left(\frac{\pi\cancel{\alpha^4}}{\alpha^{\cancel{5}}}\right.\right.$$

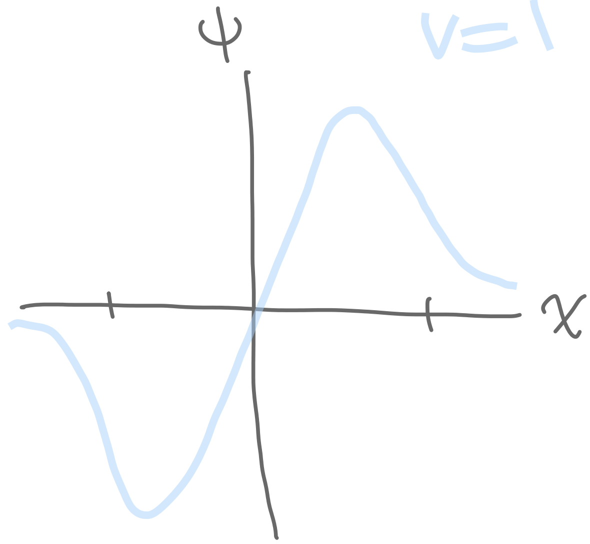
$$1 = 2\cdot|C_0|^2\left(\frac{\pi}{\ }\right)^{\frac{1}{2}}$$

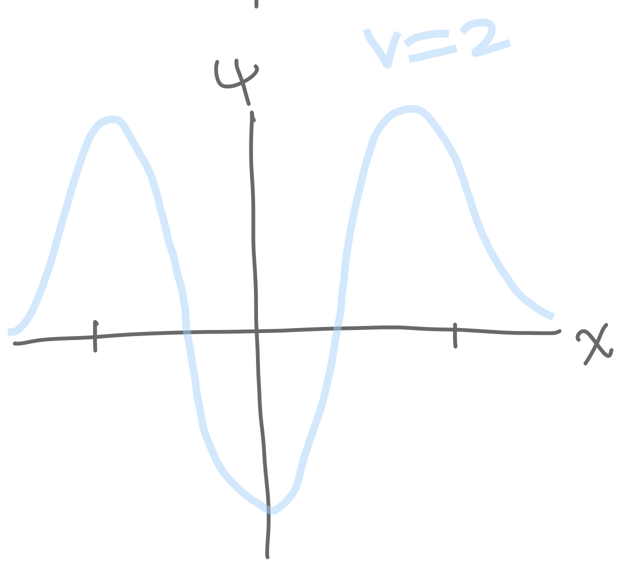
$$1 = 2\cdot|C_0|\left(\frac{}{\alpha}\right)$$

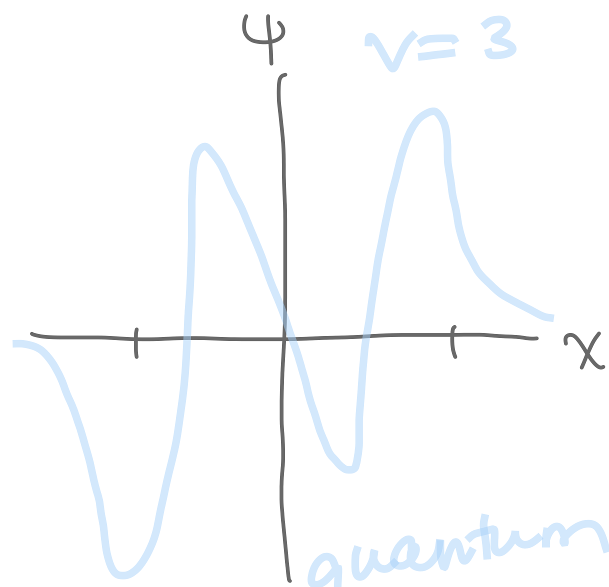
$$|C_0| = \left(\frac{\alpha}{4\pi}\right)^{\frac{1}{4}}$$

$$\boxed{\psi_2 = \left(\frac{\alpha}{4\pi}\right)^{\frac{1}{4}}(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}}$$

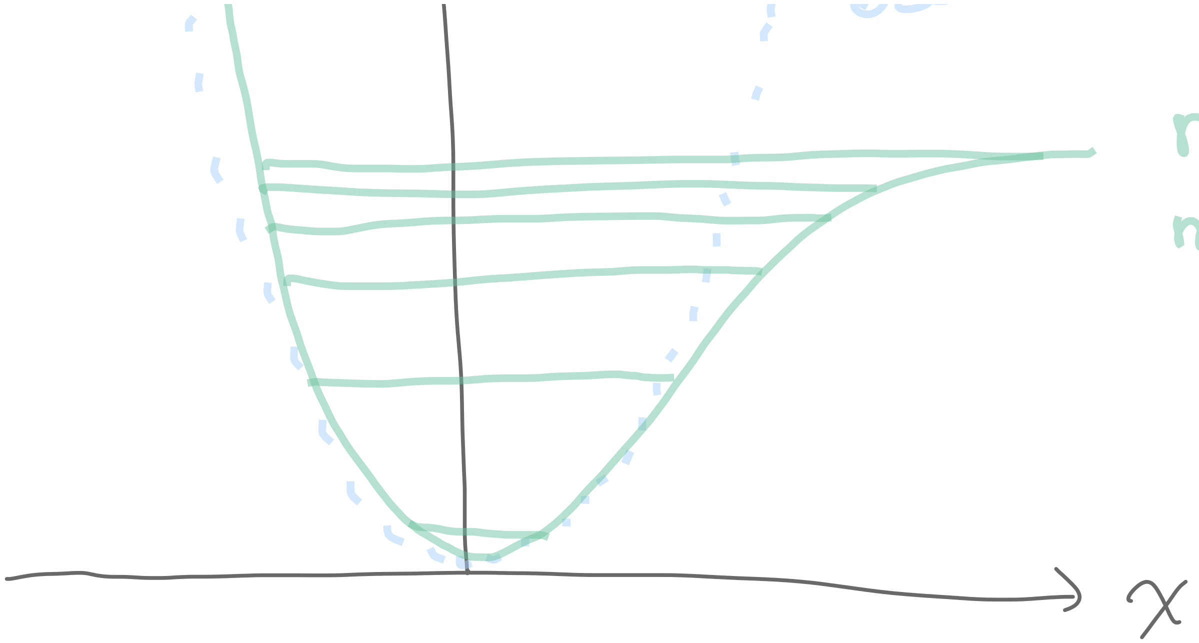








quantum harmonic oscillator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