# Lecture 9

Tuesday, October 8, 2024 11:40

# Chapter 5 : Angular Momentum

## 5.1 Simultaneous Specification of Several Properties

If two operators commute with each other, then there can exist a complete set of simultaneous eigenfunctions for these operators.

Recall that $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

① $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

$$= -\hat{B}\hat{A} + \hat{A}\hat{B}$$

$$= -(\hat{B}\hat{A} - \hat{A}\hat{B})$$

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$

② $[\hat{A}, \hat{A}^n] = \hat{A}\hat{A}^n - \hat{A}^n\hat{A}$

$$= \hat{A}^{n+1} - \hat{A}^{n+1}$$

$$[\hat{A}, \hat{A}^n] = 0, \quad n = 1, 2, 3, \ldots$$

③ $$[k\hat{A}, \hat{B}] = (k\hat{A})\hat{B} - \hat{B}(k\hat{A})$$

$$= k\hat{A}\hat{B} - k\hat{B}\hat{A}$$

linear operators in QM

$$= k(\hat{A}\hat{B} - \hat{B}\hat{A})$$

$$= \hat{A}(k\hat{B}) - (k\hat{B})\hat{A}$$

$$[k\hat{A}, \hat{B}] = [\hat{A}, k\hat{B}] = k[\hat{A}, \hat{B}]$$

4a $$[\hat{A}, \hat{B} + \hat{C}] = \hat{A}(\hat{B} + \hat{C}) - (\hat{B} + \hat{C})\hat{A}$$

$$= \hat{A}\hat{B} + \hat{A}\hat{C} - \hat{B}\hat{A} - \hat{C}\hat{A}$$

$$= (\hat{A}\hat{B} - \hat{B}\hat{A}) + (\hat{A}\hat{C} - \hat{C}\hat{A})$$

$$[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$$

4b $$[\hat{A} + \hat{B}, \hat{C}] = (\hat{A} + \hat{B})\hat{C} - \hat{C}(\hat{A} + \hat{B})$$

$$= \hat{A}\hat{C} + \hat{B}\hat{C} - \hat{C}\hat{A} - \hat{C}\hat{B}$$

$$= (\hat{A}\hat{C} - \hat{C}\hat{A}) + (\hat{B}\hat{C} - \hat{C}\hat{B})$$

$$=(\hat{A}\hat{C}-\hat{C}\hat{A})+(\hat{B}\hat{C}-\hat{C}\hat{B})$$

$$[\hat{A}+\hat{B},\hat{C}]=[\hat{A},\hat{C}]+[\hat{B},\hat{C}]$$

5a $$[\hat{A},\hat{B}\hat{C}]=\hat{A}\hat{B}\hat{C}-\hat{B}\hat{C}\hat{A}$$

$$=\hat{A}\hat{B}\hat{C}+\hat{B}\hat{A}\hat{C}-\hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A}$$

$$=(\hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C})+(\hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A})$$

$$[\hat{A},\hat{B}\hat{C}]=[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]$$

5b $$[\hat{A}\hat{B},\hat{C}]=\hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B}$$

$$=\hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B}$$

$$=\hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B})+(\hat{A}\hat{C}-\hat{C}\hat{A})\hat{B}$$

$$[\hat{A}\hat{B},\hat{C}]=\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}$$

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Example: a) find $[\hat{x},\hat{p}_x]$

$$\left[\frac{\partial}{\partial x},\hat{x}\right]f=\frac{\partial}{\partial x}\hat{x}f-\hat{x}\frac{\partial}{\partial x}f$$

$$=f+xf'-xf'$$

$$=f$$

$$\left[\frac{\partial}{\partial x},\hat{x}\right]=1$$

$$[\hat{x}, \hat{p}_x] = [\hat{x}, \frac{\hbar}{i}\frac{\partial}{\partial x}]$$

$$= \frac{\hbar}{i}[\hat{x}, \frac{\partial}{\partial x}]$$

$$= -\frac{\hbar}{i}[\frac{\partial}{\partial x}, \hat{x}]$$

$$\therefore [\hat{x}, \hat{p}_x] = -\frac{\hbar}{i}$$

b) Find $[\hat{x}, \hat{p}_x^2]$

$$[\hat{x}, \hat{p}_x^2] = \hat{p}_x[\hat{x}, \hat{p}_x] + [\hat{x}, \hat{p}_x]\hat{p}_x$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial x}\left(-\frac{\hbar}{i}\right) + \left(-\frac{\hbar}{i}\right)\frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\therefore [\hat{x}, \hat{p}_x^2] = 2\hbar^2\frac{\partial}{\partial x}$$

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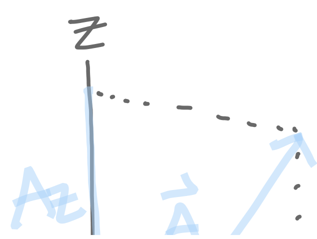
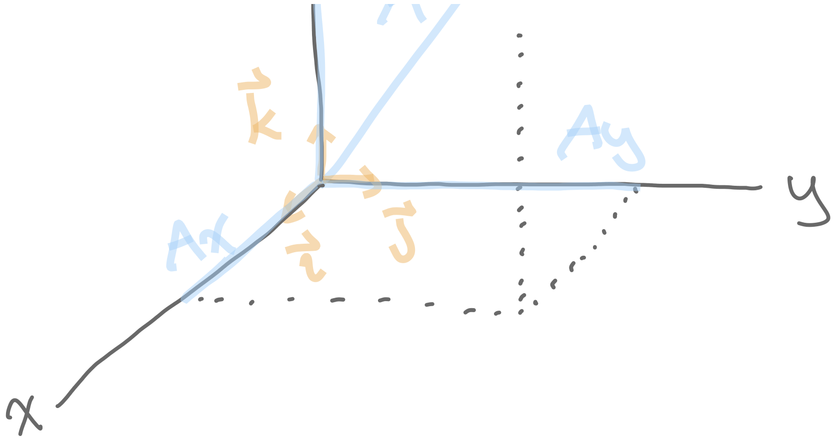
## 5.2 Vectors






$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$$

Dot Product:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta = \vec{B} \cdot \vec{A}$$

$$\vec{A} \cdot \vec{B} = (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \cdot (B_x \vec{i} + B_y \vec{j} + B_z \vec{k})$$

$$\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z}$$

$$\vec{A} \cdot \vec{A} = |\vec{A}|^2$$

$$|\vec{A}| = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}}$$

Cross Product:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$$

$$B \times A = -A \times B$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \vec{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \vec{j}$$

$$+ \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \vec{k}$$

$$\boxed{\begin{aligned} \vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\vec{i} + (A_z B_x - A_x B_z)\vec{j} \\ + (A_x B_y - A_y B_x)\vec{k} \end{aligned}}$$

$$\boxed{\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}}$$

$$\text{grad } g(x,y,z) \equiv \nabla g(x,y,z)$$

$$\equiv \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}$$

$$\vec{F} = -\nabla V(x,y,z) = -\vec{i}\frac{\partial V}{\partial x} - \vec{j}\frac{\partial V}{\partial y} - \vec{k}\frac{\partial V}{\partial z}$$

All can be generalized to n dimensions.

## 5.3 Angular Momentum of a One-Particle System

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\vec{p} \equiv m\vec{v}$$

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

$$\vec{L} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \end{vmatrix}$$

$$\begin{vmatrix} p_x & p_y & p_z \end{vmatrix}$$

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$ torque, $\vec{\tau}$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

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## One-Particle Orbital-Angular-Momentum Operators

Orbital Angular Momentum : motion of a particle through space, analogue of $\vec{L}$

Spin Angular Momentum : intrinsic to many microscopic particles and has no classical analogue

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\hat{L}^2 = |\hat{L}|^2 = \hat{L}\cdot\hat{L} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\hat{L}_y f = -i\hbar\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)$$

$$\hat{L}_x\hat{L}_y f = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\cdot -i\hbar\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)$$

$$= -\hbar^2\left(y\frac{\partial}{\partial z}z\frac{\partial f}{\partial x} - y\frac{\partial}{\partial z}x\frac{\partial f}{\partial z}\right.$$

$$\left. - z\frac{\partial}{\partial y}z\frac{\partial f}{\partial x} + z\frac{\partial}{\partial y}x\frac{\partial f}{\partial z}\right)$$

$$\hat{L}_x\hat{L}_y f = -\hbar^2\left(y\frac{\partial f}{\partial x} + \cancel{yz\frac{\partial^2 f}{\partial z\partial x}} - yx\cancel{\frac{\partial^2 f}{\partial z^2}}\right.$$

$$\boxed{-z^2\cancel{\frac{\partial^2 f}{\partial y \partial x}} + zx\cancel{\frac{\partial^2 f}{\partial y \partial z}}\Big)}$$

$$\hat{L}_x f = -i\hbar\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)$$

$$\hat{L}_y\hat{L}_x f = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \cdot -i\hbar\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)$$

$$= -\hbar^2\Big(z\frac{\partial}{\partial x}y\frac{\partial f}{\partial z} - z\frac{\partial}{\partial x}z\frac{\partial f}{\partial y} - x\frac{\partial}{\partial z}y\frac{\partial f}{\partial z} + x\frac{\partial}{\partial z}z\frac{\partial f}{\partial y}\Big)$$

$$\boxed{\hat{L}_y\hat{L}_x f = -\hbar^2\Big(zy\cancel{\frac{\partial^2 f}{\partial x \partial z}} - z^2\cancel{\frac{\partial^2 f}{\partial x \partial y}} - xy\cancel{\frac{\partial^2 f}{\partial z^2}} + x\frac{\partial f}{\partial y} + xz\cancel{\frac{\partial^2 f}{\partial z \partial y}}\Big)}$$

$$\hat{L}_x\hat{L}_y f - \hat{L}_y\hat{L}_x f = -\hbar^2\left(y\frac{\partial f}{\partial x} - x\right.$$

$$\boxed{[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z}$$