Lecture 10

Monday, October 16, 2023 15:41

1. Imagine a particle confined to move on the circumference of a circle, i.e. "a particle on a ring", such that its position can be described by an angle ϕ in the range 0 to 2π. Find the normalizing factor for the following wavefunctions:

   a) $\cos\phi$

   b) $e^{im_l\phi}$, where m_l is an integer

a) $\psi = \cos\phi$

$$N^2 \int |\psi|^2 = 1$$

$$N^2 \int_0^{2\pi} \cos^2 \phi \, d\phi = 1$$

Integral table:

$$\int \cos^2 kx \, dx = \frac{x}{2} + \frac{1}{4k} \sin 2kx$$

$$N^2 \left[\frac{\phi}{2} + \frac{1}{4} \underset{0}{\cancel{\sin 2\phi}} \right]_0^{2\pi} = 1$$

$$N^2 \pi = 1$$

$$N = \frac{1}{\sqrt{\pi}}$$

Normalized wave function:

$$\psi = \frac{1}{\sqrt{\pi}} \cos \phi$$

b) $\psi = e^{i m_l \phi}$

$$\int \psi^* \psi \, dZ = 1$$

$$N^2 \int_0^{2\pi} e^{\underline{-im_\ell \phi}} \, e^{\underline{im_\ell \phi}} \, d\phi = 1$$

$$N^2 \int_0^{2\pi} d\phi = 1$$

$$N^2 (2\pi) = 1$$

$$N = \frac{1}{\sqrt{2\pi}}$$

Normalized wavefunction:

$$\psi = \frac{1}{\sqrt{2\pi}} e^{im_\ell \phi}$$

2. Identify which of the following functions are eigenfunctions of the inversion operator $\hat{i}$, which has the effect of making the replacement $x \rightarrow -x$.

a) $x^3 - kx$

b) $\cos kx$

c) $x^2 + 3x - 1$

Identify the eigenvalue of $\hat{\imath}$ when relevant.

a) $\hat{\imath}\,(x^3 - kx)$

$$= (-x)^3 - k(-x)$$

$$= -x^3 + kx$$

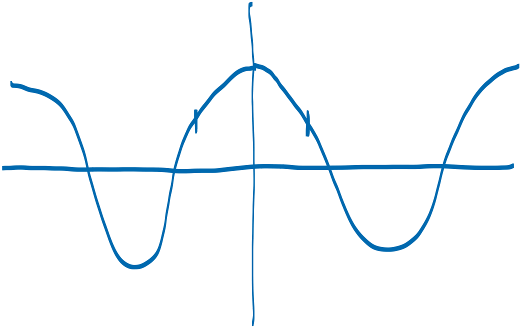
$$= -(x^3 - kx)$$

$\therefore$ $x^3 - kx$ is an eigenfunction of $\hat{\imath}$, with eigenvalue -1.

b) $\hat{\imath}\,(\cos kx)$

$$= \cos[k(-x)]$$

$$= \cos kx$$



$\therefore$ $\cos kx$ is an eigenfunction of $\hat{\imath}$, with eigenvalue $+1$.

c) $\hat{\imath}\,(x^2 + 3x - 1)$

$= (-x)^2 + 3(-x) - 1$

$= x^2 - 3x - 1$

$\therefore$ $x^2 + 3x - 1$ is not an eigenfunction of $\hat{\imath}$.

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3. A particle freely moving in one dimension x with $0 \le x \le \infty$ is in a

State described by the normalized wave-function

$$\psi(x) = a^{1/2} e^{-ax/2}$$

where a is a constant. Evaluate the expectation value of the position operator.

$$\langle B \rangle = \int \psi^* \hat{B} \psi \, d\tau$$

$$\langle x \rangle = \int_0^\infty \psi \hat{B} \psi \, dx$$

$$= a \int_0^\infty e^{-ax/2} \hat{x} e^{-ax/2} \, dx$$

$$\boxed{\hat{x} \equiv x.}$$

$$= a \int_0^\infty x e^{-ax} \, dx$$

Integral table :

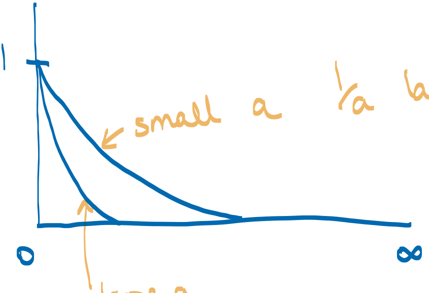
$$\int x e^{-kx}\,dx = -\frac{1}{k^2}e^{-kx} - \frac{x}{k}e^{-kx}$$

$$= a\left[-\frac{1}{a^2}e^{-ax} - \frac{x}{a}e^{-ax}\right]_0^{\infty}$$

$$= a\left[0-\left(-\frac{1}{a^2}\right)\right]$$

$$\langle x\rangle = \frac{1}{a}$$

$$\psi(x) = a^{1/2}e^{-ax/2}$$



small a $1/a$ large $\langle x\rangle$ large

large a

$1/a$ small

$\langle x\rangle$ small

4. Evaluate the commutators of the operators

a) $\frac{d}{dx}$ and $\frac{1}{x}$

b) $\frac{d}{dx}$ and x^2

commutator : $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

a) $[\frac{d}{dx}, \frac{1}{x}] = \frac{d}{dx}\frac{1}{x} - \frac{1}{x}\frac{d}{dx}$

$$\left[\frac{d}{dx}, \frac{1}{x}\right] f(x) = \frac{d}{dx}\left[\frac{1}{x} f(x)\right] - \frac{1}{x}\left[\frac{d}{dx} f(x)\right]$$

$$= \cancel{\frac{1}{x} f'(x)} + \left[-\frac{1}{x^2} f(x)\right]$$

$$- \cancel{\frac{1}{x} f'(x)}$$

$$\left[\frac{d}{dx}, \frac{1}{x}\right] f(x) = -\frac{1}{x^2} f(x)$$

$$\therefore \left[\frac{d}{dx}, \frac{1}{x}\right] = -\frac{1}{x^2}$$

b) $$\left[\frac{d}{dx}, x^2\right] = \boxed{\frac{d}{dx}x^2 - x^2\frac{d}{dx}}$$

$$\left[\frac{d}{dx}, x^2\right]f(x) = \underbrace{\frac{d}{dx}\left[x^2 f(x)\right]} - x^2\left[\frac{d}{dx}f(x)\right]$$

$$= 2xf(x) + \cancel{x^2 f'(x)}$$

$$- \cancel{x^2 f'(x)}$$

$$\left[\frac{d}{dx}, x^2\right]f(x) = 2xf(x)$$

$$\therefore \left[\frac{d}{dx}, x^2\right] = 2x$$

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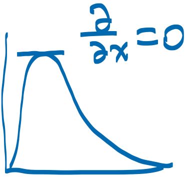
5. For the $v=0,1$ harmonic oscillator

state, find the most likely position(s) of the particle.



$$\psi_0 = c_0 e^{-\alpha x^2/2}$$

$$\psi_1 = c_1 x e^{-\alpha x^2/2}$$

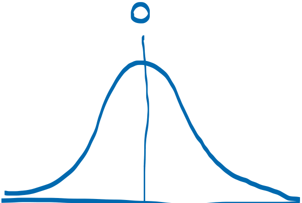
$$\frac{\partial}{\partial x} |\psi|^2 = 0$$

$$\frac{\partial}{\partial x} \; c_0^2 e^{-\alpha x^2}$$

$$= c_0^2 \; e^{-\alpha x^2} \cdot -\alpha \cdot 2x$$

$$= -2\alpha x c_0^2 \underbrace{e^{-\alpha x^2}}_{} = 0$$

$x = 0$



$$\psi_1 = c_1 x e^{-\alpha x^2/2}$$

$$\frac{\partial}{\partial x} |\psi_1|^2 = \frac{\partial}{\partial x} \; c_1^2 x^2 e^{-\alpha x^2}$$

$$= c_1^2 \Big(2x e^{-\alpha x^2}$$

$$+ \; x^2 e^{-\alpha x^2} \cdot -\alpha \cdot 2x \Big)$$

$$= 2c_1^2 e^{-\alpha x^2} (x - \alpha x^3) = 0$$

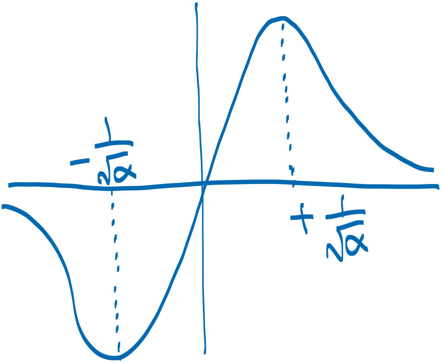
$$x - \alpha x^3 = 0$$

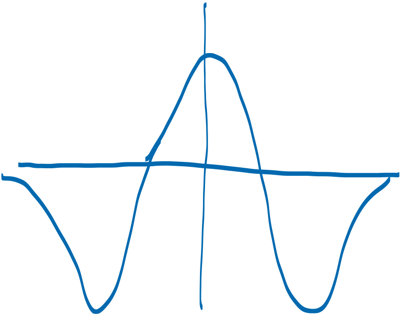
$$x(1 - \alpha x^2) = 0$$

$$\cancel{x = 0}$$

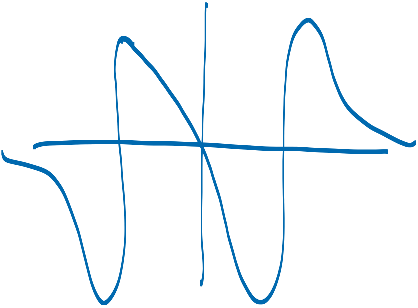
$1 - \alpha x^2 = 0$

$x = \pm \frac{1}{\sqrt{\alpha}}$





$v = 2$



$v = 3$