Lecture 11

Monday, October 23, 2023 15:53

Deferred exam; will grade by next Monday October 30; also review solutions

Chapter 5 : Angular Momentum

If $[\hat{A}, \hat{B}] = 0$, then ψ can be an eigenfunction of both $\hat{A}$ and $\hat{B}$.

If $[\hat{A}, \hat{B}] \neq 0$, then we cannot determine A and B (with certainty) simultaneously.

$[\hat{x}, \hat{p}_x] \neq 0$, so we cannot determine both position and momentum (precisely) simultaneously.

↳ Heisenberg Uncertainty Principle

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$

$$\boxed{[\hat{A}, \hat{A}^n] = 0, \quad n = 1, 2, 3, \ldots}$$

$$[k\hat{A},\hat{B}] = [\hat{A}, k\hat{B}] = k[\hat{A},\hat{B}]$$

$$[\hat{A},\hat{B}+\hat{C}] = [\hat{A},\hat{B}] + [\hat{A},\hat{C}]$$

$$[\hat{A}+\hat{B},\hat{C}] = [\hat{A},\hat{C}] + [\hat{B},\hat{C}]$$

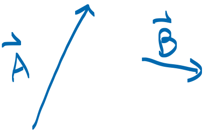
$$[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$$

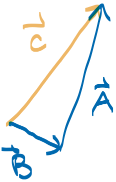
$$\boxed{[\hat{A}\hat{B},\hat{C}] = [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}]}$$

k is a constant ; all operators are linear

5.2 Vectors

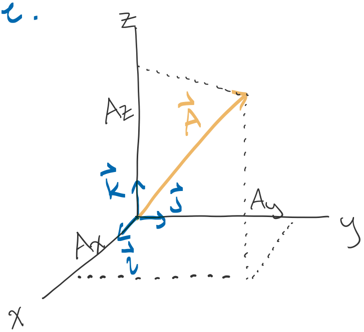
- scalars : completely specified by their magnitude
- vectors : need both magnitude and direction





$$\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

$c\vec{A}$: a vector of length $|c|$ times the length of $\vec{A}$ with the same direction as $\vec{A}$ if c is positive, or the opposite direction to $\vec{A}$ if c is negative.



$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$$ same as (A_x, A_y, A_z)

$$\vec{A} + \vec{B} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k} + B_x\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k}$$

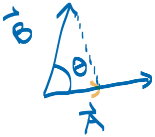
$$\vec{A} + \vec{B} = (A_x + B_x)\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_z)\vec{k}$$

$$c\vec{A} = cA_x\vec{\imath} + cA_y\vec{\jmath} + cA_z\vec{k}$$

$|\vec{A}|$: magnitude or length of $\vec{A}$; scalar

Dot product

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = \vec{B}\cdot\vec{A}$$



projection of $\vec{B}$ on $\vec{A}$:

$|\vec{B}|\cos\theta$

$$\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = \cos 0 = 1$$

$$\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\vec{A}\cdot\vec{B} = \left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right)\cdot\left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right)$$

$$= A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2$$

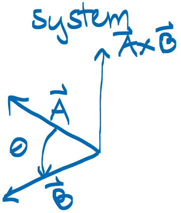
$$= |\vec{A}|^2$$

$$|\vec{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$$

For three-dimensional vectors, there is also the cross product

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$$

whose line segment is perpendicular to the plane defined by $\vec{A}$ and $\vec{B}$, and whose direction is such that $\vec{A}$, $\vec{B}$, and $\vec{A}\times\vec{B}$ form a right-handed system (counterclockwise)



$$\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$$

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \sin 0 = 0$$

$$\vec{i}\times\vec{j} = \vec{k}, \quad \vec{j}\times\vec{i} = -\vec{k}$$

$$\vec{j}\times\vec{k} = \vec{i}, \quad \vec{k}\times\vec{j} = -\vec{i}$$

$$\vec{k}\times\vec{i} = \vec{j}, \quad \vec{i}\times\vec{k} = -\vec{j}$$

$$\vec{A}\times\vec{B} = (A_x\vec{i} + A_y\vec{j} + A_z\vec{k})$$

$$\times\left(B_x\vec{i}+B_y\vec{j}+B_z\vec{k}\right)$$

$$=\cancel{A_xB_x\vec{i}\times\vec{i}}+A_xB_y\overset{\vec{k}}{\vec{i}\times\vec{j}}+A_xB_z\overset{-\vec{j}}{\vec{i}\times\vec{k}}$$

$$+A_yB_x\overset{-\vec{k}}{\vec{j}\times\vec{i}}+\cancel{A_yB_y\vec{j}\times\vec{j}}+A_yB_z\overset{\vec{i}}{\vec{j}\times\vec{k}}$$

$$+A_zB_x\underset{\vec{j}}{\vec{k}\times\vec{i}}+A_zB_y\underset{-\vec{i}}{\vec{k}\times\vec{j}}+\cancel{A_zB_z\vec{k}\times\vec{k}}$$

$$=\left(A_xB_y-A_yB_x\right)\vec{k}$$

$$+\left(A_zB_x-A_xB_z\right)\vec{j}$$

$$+\left(A_yB_z-A_zB_y\right)\vec{i}$$

$$\boxed{\begin{aligned}\vec{A}\times\vec{B}=&\left(A_yB_z-A_zB_y\right)\vec{i}+\left(A_zB_x-A_xB_z\right)\vec{j}\\&+\left(A_xB_y-A_yB_x\right)\vec{k}\end{aligned}}$$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}$$

$$=\vec{i}\begin{vmatrix}A_y&A_z\\B_y&B_z\end{vmatrix}-\vec{j}\begin{vmatrix}A_x&A_z\\B_x&B_z\end{vmatrix}$$

$$+\vec{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= (A_yB_z - A_zB_y)\vec{i} + (A_zB_x - A_xB_z)\vec{j}$$

$$+ (A_xB_y - A_yB_x)\vec{k}$$

vector operator del

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\vec{p} = -i\hbar\nabla$$

gradient of a function $g(x,y,z)$ is defined as the result of operating on g with del:

$$\text{grad } g(x,y,z) \equiv \nabla g(x,y,z)$$

$$\equiv \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}$$

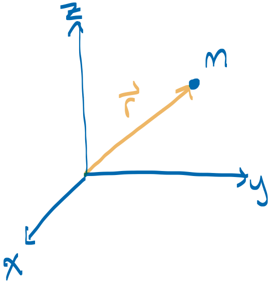
The gradient of a scalar function is a vector function.

$$\vec{F} = -\nabla V(x,y,z) = -\vec{i}\frac{\partial V}{\partial x} - \vec{j}\frac{\partial V}{\partial y} - \vec{k}\frac{\partial V}{\partial z}$$

$$A_x(t),\ A_y(t),\ A_z(t)$$

$$\frac{d\vec{A}}{dt} = \vec{i}\frac{dA_x}{dt} + \vec{j}\frac{dA_y}{dt} + \vec{k}\frac{dA_z}{dt}$$

5.3 Angular Momentum of a One-Particle System



$\vec{r}$: instantaneous position of the particle $\vec{r} = (x, y, z)$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\frac{d\vec{r}}{dt} = \vec{i}\frac{dx}{dt} + \vec{j}\frac{dy}{dt} + \vec{k}\frac{dz}{dt}$$

$$\vec{v} = \vec{i}\,v_x + \vec{j}\,v_y + \vec{k}\,v_z$$

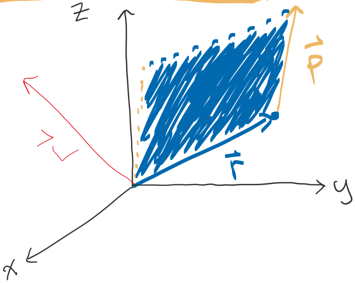
$$\vec{p} \equiv m\vec{v}$$

$$p_x = mv_x \,,\; p_y = mv_y \,,\; p_z = mv_z$$

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$ (classical mechanics)

$$\vec{L} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$\boxed{\begin{aligned} L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x \end{aligned}}$$



One-Particle Orbital Angular Momentum

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$y\frac{\partial}{\partial x} = \frac{\partial}{\partial x}y$$

$$\hat{L}^2 = |\hat{\vec{L}}|^2 = \hat{\vec{L}}\cdot\hat{\vec{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\hat{L}_y f = -i\hbar\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)$$

$$\hat{L}_x\hat{L}_y f = \left[-i\hbar\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)\right]\left[-i\hbar\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)\right]$$

$$= -\hbar^2\left(y\frac{\partial f}{\partial z}z\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial z}x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}z\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y}x\frac{\partial f}{\partial z}\right)$$

$$\boxed{\begin{aligned}\hat{L}_x\hat{L}_y f = -\hbar^2\Bigg(&\boxed{y\frac{\partial f}{\partial x}} + \cancel{yz\frac{\partial^2 f}{\partial x\partial z}} - \cancel{yx\frac{\partial^2 f}{\partial z^2}}\\ &- z^2\cancel{\frac{\partial^2 f}{\partial x\partial y}} + zx\cancel{\frac{\partial^2 f}{\partial z\partial y}}\Bigg)\end{aligned}}$$

$$\hat{L}_x f = -i\hbar\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)$$

$$\hat{L}_y \hat{L}_x f = \left[i\hbar \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]$$

$$\left[-i\hbar \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)\right]$$

$$= -\hbar^2 \left(z\frac{\partial}{\partial x} y \frac{\partial f}{\partial z} - z\frac{\partial}{\partial x} z\frac{\partial f}{\partial y}\right.$$

$$\left. - x\frac{\partial}{\partial z} y\frac{\partial f}{\partial z} + x\frac{\partial}{\partial z} z \frac{\partial f}{\partial y}\right)$$

$$\boxed{\hat{L}_y\hat{L}_x f = -\hbar^2 \left(\cancel{zy\frac{\partial^2 f}{\partial z\partial x}} - \cancel{z^2\frac{\partial^2 f}{\partial y \partial x}} - \cancel{xy\frac{\partial^2 f}{\partial z^2}} + \boxed{x\frac{\partial f}{\partial y}} + \cancel{xz\frac{\partial^2 f}{\partial y\partial z}}\right)}$$

$$[\hat{L}_x, \hat{L}_y] = \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x$$

$$[\hat{L}_x, \hat{L}_y] = -\hbar^2 \left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)$$

$$\hat{L}_z = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\hat{L}_z = i\hbar \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

$\times i\hbar$

$$\boxed{[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z}$$

$$\boxed{[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x}$$

$$\boxed{[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y}$$

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_x]$$

$$= \cancelto{0}{[\hat{L}_x^2, \hat{L}_x]} + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x]$$

$$= [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x]$$

$$= [\hat{L}_y, \hat{L}_x]\hat{L}_y + \hat{L}_y[\hat{L}_y, \hat{L}_x]$$

$$+ [\hat{L}_z, \hat{L}_x]\hat{L}_z + \hat{L}_z[\hat{L}_z, \hat{L}_x]$$

$$= -i\hbar \hat{L}_z\hat{L}_y - i\hbar \hat{L}_y\hat{L}_z$$

$$+ i\hbar \hat{L}_y\hat{L}_z + i\hbar \hat{L}_z\hat{L}_y$$

$$\boxed{[\hat{L}^2, \hat{L}_x] = 0}$$