Lecture 13

Monday, October 30, 2023 15:58

no curve!

Midterm Exam Solutions

1. True or False

a) The ground state energy of a particle in a box is zero.

$$E = \frac{n^2 h^2}{8 m l^2}, \quad n = 1, 2, 3, \ldots$$

False

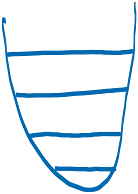
b) If we measure the property B when the system's state function is not an eigenfunction of $\hat{B}$, then we can get a result that is not an eigenvalue of $\hat{B}$.

False Always get an eigenvalue of $\hat{B}$

When eigenfunction : b

When not eigenfunction : $\int \psi^* \hat{B} \psi \, dz$

c) The harmonic oscillator energy levels are equally spaced.



$$E = (v + \tfrac{1}{2}) h\nu,$$

$$v = 0, 1, 2, \ldots$$

True

d) If g is an eigenfunction of the linear operator $\hat{B}$, then cg is also an eigenfunction of $\hat{B}$, where c is an arbitrary constant.

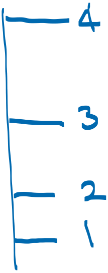
$$\hat{A}\,[c f(x)] = c \hat{A} f(x)$$

True

e) A probability density can never be negative.

True

f) The particle in a box transition that absorbs the photon of shortest wavelength is from the n=1 to the n=2 level.



$$E = \frac{n^2 h^2}{8 m l^2}$$

False

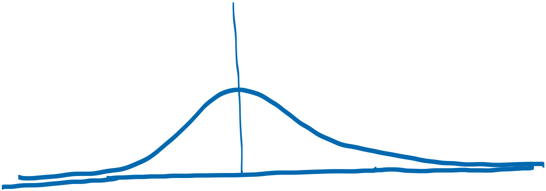
(i) e^{-ax^2}

(ii) e^{-ax}

$-\infty < x < +\infty$

$$N^2\int_{-\infty}^{\infty} |\psi|^2 \, dx$$

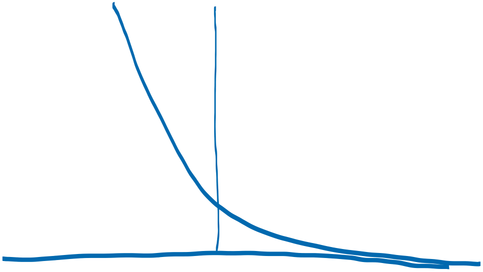
(ii) $$N^2\int_{-\infty}^{\infty} e^{-2ax^2} \, dx$$



can be normalized; function does not go to infinity at any point

(ii) $$N^2\int_{-\infty}^{\infty} |\psi|^2 \, dx$$

$$= N^2 \int_{-\infty}^{\infty} e^{-2ax} \, dx$$



function approaches ∞ when $x \to -\infty$. Not normalizable.

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3. a) Solve the following differential equation

$$2y''(x) - 10y'(x) + 12y(x) = 0$$

$$2a^2 - 10a + 12 = 0$$

$$a^2 - 5a + 6 = 0$$

$a = +2, +3$

$$\boxed{y(x) = c_1 e^{2x} + c_2 e^{3x}}$$

b) Evaluate the arbitrary constants in the solution if the boundary conditions are

$y = 0$ at $x = 0$

$y' = 1$ at $x = 0$

$$y(0) = c_1 e^{2(0)} + c_2 e^{3(0)} = 0$$

$$c_1 + c_2 = 0$$

$$y'(0) = 2c_1 e^{2(0)} + 3c_2 e^{3(0)} = 1$$

$$2c_1 + 3c_2 = 1$$

$$c_1 = -1, \quad c_2 = +1$$

$$\boxed{y(x) = e^{3x} - e^{2x}}$$

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4. Calculate the zero-point energy of a harmonic oscillator consisting of a particle of mass 5.16×10^{-26} kg and force constant $285\ \mathrm{N \cdot m^{-1}}$.

$$E = \left(v + \frac{1}{2}\right) h\nu, \quad v = \boxed{0}, 1, 2, \ldots$$

$$\nu = \frac{1}{2\pi}\left(\frac{k}{m}\right)^{1/2}$$

$$E = \left(0 + \frac{1}{2}\right)\left(\frac{1}{2\pi}\right) 6.62 \times 10^{-34}\ \mathrm{J \cdot Hz^{-1}} \cdot \left(\frac{285\ \mathrm{kg \cdot s^{-2}}}{5.16 \times 10^{-26}\ \mathrm{kg}}\right)^{1/2}$$

$$\boxed{E = 3.92 \times 10^{-21}\ \text{J}}$$

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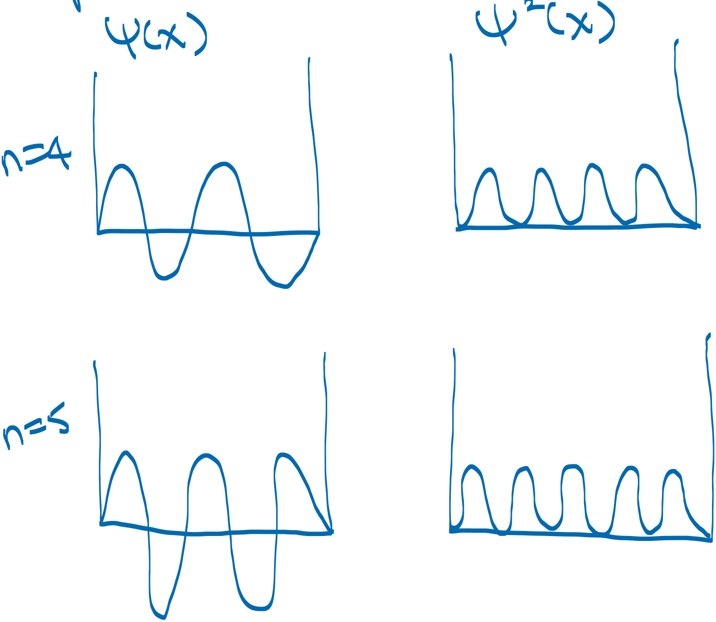
5. $\psi(x)$ $\psi^2(x)$ $n=4$, $n=5$

particle in a box states



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6. Identify which of the following

functions are eigenfunctions of the operator $\frac{d}{dx}$:

(i) $\cos(kx)$

$$\frac{d}{dx}\cos(kx) = -k\sin(kx)$$

not an eigenfunction

(ii) e^{ikx}

$$\frac{d}{dx}e^{ikx} = \boxed{ik}\,e^{ikx}$$

eigenfunction, with eigenvalue ik

(iii) kx

$$\frac{d}{dx}kx = k$$

not an eigenfunction

(iv) e^{-ax^2}

$$\frac{d}{dx}e^{-ax^2} = -2ax\,e^{-ax^2}$$

$$\boxed{\text{not an eigenfunction}}$$

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7. Calculate the expectation value of the linear momentum, $\langle p_x \rangle$, of a particle described by the following normalized wavefunctions:

a) Ne^{ikx}

b) $N\cos(kx)$

(No need to find N)

a) Ne^{ikx}

$$\langle p_x \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p}_x \psi \, dx$$

$$= N^2 \int_{-\infty}^{\infty} e^{-ikx} \frac{\hbar}{i} \frac{d}{dx} e^{ikx} \, dx$$

$$= N^2 \hbar k \int_{-\infty}^{\infty} e^{-ikx} e^{ikx} \, dx$$

$$= \xcancel{N^2 \hbar k (2\infty)}$$

$$\hat{p}_x \psi = \frac{\hbar}{i} \frac{d}{dx} N e^{ikx}$$

$$= \boxed{\hbar k} N e^{ikx}$$

$$\boxed{\therefore \langle p_x \rangle = \hbar k}$$

b) $$\langle p_x \rangle = N^2 \int_{-\infty}^{\infty} \cos(kx) \, \hat{p}_x \cos(kx) \, dx$$

$$= N^2 \int_{-\infty}^{\infty} \cos(kx) \frac{\hbar}{i} \frac{d}{dx} [\cos(kx)] \, dx$$

$$= -\frac{\hbar k N^2}{i} \underbrace{\int_{-\infty}^{\infty} \cos(kx)\sin(kx)\,dx}_{0\text{, on equation sheet}}$$

$$= 0$$

$$\boxed{\langle p_x \rangle = 0}$$

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8. Evaluate the following commutators:

a) $[\hat{x}, \hat{p}_x]$

b) $[\hat{x}, \hat{p}_x^{\,2}]$

c) $[\hat{x}, \hat{p}_y]$

a) $[\hat{x}, \hat{p}_x] f = \hat{x}\hat{p}_x f - \hat{p}_x \hat{x} f$

$$= x\frac{\hbar}{i}\frac{\partial}{\partial x} f - \frac{\hbar}{i}\frac{\partial}{\partial x}(x f)$$

$$= \cancel{\frac{\hbar}{i} x \frac{\partial f}{\partial x}} - \frac{\hbar}{i} f - \cancel{\frac{\hbar}{i}\frac{\partial f}{\partial x}}$$

$$= -\frac{\hbar}{i} f$$

$$\boxed{[\hat{x}, \hat{p}_x] = -\frac{\hbar}{i}}$$

b) $[x, \hat{p}_x^2]f = \hat{x}\hat{p}_x^2 f - \hat{p}_x^2 \hat{x} f$

$$= x\frac{\hbar^2}{i^2}\frac{\partial^2}{\partial x^2} f - \frac{\hbar^2}{i^2}\frac{\partial^2}{\partial x^2}(xf)$$

$$= \cancel{\frac{\hbar^2}{i^2} x \frac{\partial^2 f}{\partial x^2}} - \frac{\hbar^2}{i^2}\frac{\partial f}{\partial x}$$

$$- \frac{\hbar^2}{i^2}\frac{\partial f}{\partial x} - \cancel{\frac{\hbar^2}{i^2} x \frac{\partial^2 f}{\partial x^2}}$$

$$= -\frac{2\hbar^2}{i^2}\frac{\partial f}{\partial x}$$

$$\boxed{[\hat{x}, \hat{p}_x^2] = 2\hbar^2 \frac{\partial}{\partial x}}$$

c) $[\hat{x}, \hat{p}_y]f = \hat{x}\hat{p}_y f - \hat{p}_y \hat{x} f$

$$= x\frac{\hbar}{i}\frac{\partial}{\partial y}f - \frac{\hbar}{i}\frac{\partial}{\partial y}(xf)$$

$$= \cancel{x\frac{\hbar}{i}\frac{\partial f}{\partial y}} - \cancel{x\frac{\hbar}{i}\frac{\partial f}{\partial y}}$$

$$= 0$$

$$\boxed{[\hat{x}, \hat{p}_y] = 0}$$

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9. For the particle in a cubic box, what is the degree of degeneracy of the energy levels with the following values of $E(8ma^2/h^2)$?

$$n_x^2 + n_y^2 + n_z^2 = E\left(\frac{8ma^2}{h^2}\right)$$

(a) 12 ← $4+4+4$ ψ_{222} $\boxed{1}$

(b) 14

(c) 27

$1, 4, 9, 16, 25, \ldots$

$14 = 9 + 4 + 1$

$3, 2, 1$

$\psi_{123}, \psi_{132}, \psi_{213}, \psi_{231}, \psi_{312}, \psi_{321}$

$\boxed{6}$

(c) $27 = 9 + 9 + 9$ $\quad \psi_{333}$

$27 = 25 + 1 + 1$

$5 \quad 1 \quad 1$

$\psi_{511}, \psi_{151}, \psi_{115}$ $\quad \boxed{4}$

$$T(\phi) = A e^{im\phi}, \quad \underbrace{m = 0, \pm 1, \pm 2, \ldots}_{\text{eigenvalues are quantized}}$$

eigenfunction of $\hat{L}_z$

Need to normalize $T(\phi)$.

$0 \leq r \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$

$$\boxed{d\tau = r^2 dr \sin\theta \, d\theta \, d\phi}$$

volume element

$$\int_0^\infty \left[\int_0^\pi \left[\int_0^{2\pi} |F(r,\theta,\phi)|^2 d\phi \right] \sin\theta \, d\theta \right] r^2 dr = 1$$

$$F(r,\theta,\phi) = R(r)S(\theta)T(\phi)$$

$$\int_0^\infty |R|^2 r^2 dr = 1, \quad \int_0^\pi |S|^2 \sin\theta \, d\theta = 1$$

$$\int_0^{2\pi} |T|^2 d\phi = 1$$

$$\int_0^{2\pi} \left(Ae^{im\phi}\right)^* Ae^{im\phi} = 1$$

2π

$$A^2 \int_0 d\phi = 1$$

$$|A| = \frac{1}{\sqrt{2\pi}}$$

$$\boxed{T(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad , \; m = 0, \pm 1, \pm 2, \ldots}$$