Lecture 15

Monday, November 6, 2023 15:56

November 27 LDC is away
Shayne will give lecture

November 29 exam review (during class time)

December 1 puzzle challenge
(some amount of bonus points)

December 5 08:30 – 10:30
Final Exam cumulative, likely with heavier emphasis on second half of the course

Ladder Operators Continued

$$[\hat{M}_x, \hat{M}_y] = i\hbar \hat{M}_z$$
$$[\hat{M}_y, \hat{M}_z] = i\hbar \hat{M}_x$$
$$[\hat{M}_z, \hat{M}_x] = i\hbar \hat{M}_y$$

$\hat{M}_+ \equiv \hat{M}_x + i\hat{M}_y$ raising

$\hat{M}_- \equiv \hat{M}_x - i\hat{M}_y$ lowering

$$\hat{M}_+\hat{M}_z = \boxed{\hat{M}_z\hat{M}_+ - \hbar\hat{M}_+}$$

$$\hat{M}_-\hat{M}_z = \hat{M}_z\hat{M}_- + \hbar\hat{M}_-$$

derived previously

$$\hat{M}^2 Y = cY$$

$$\hat{M}_z Y = bY$$

apply $\hat{M}_+$

$$\boxed{\hat{M}_+\hat{M}_z} Y = \hat{M}_+ bY$$

$$(\hat{M}_z\hat{M}_+ - \hbar\hat{M}_+)Y = b\hat{M}_+ Y$$

$$\hat{M}_z(\hat{M}_+ Y) = (b+\hbar)(\hat{M}_+ Y)$$

apply $\hat{M}_+$ again

$$\boxed{\hat{M}_+\hat{M}_z}\hat{M}_+ Y = \hat{M}_+(b+\hbar)\hat{M}_+ Y$$

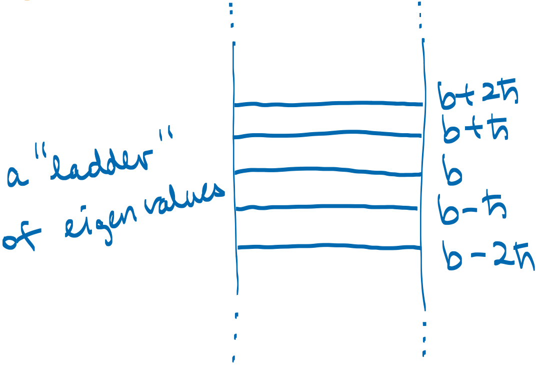
$$(\hat{M}_z\hat{M}_+ - \underbrace{\hbar\hat{M}_+)\hat{M}_+ Y}_{} = (b+\hbar)\hat{M}_+^2 Y$$

$$\hat{M}_z(\hat{M}_+^2 Y) = (b+2\hbar)(\hat{M}_+^2 Y)$$

$$\boxed{\begin{gathered}\hat{M}_z(\hat{M}_+^k Y) = (b + k\hbar)(\hat{M}_+^k Y) \\ k = 0, 1, 2, \ldots\end{gathered}}$$

$$\hat{M}_z(\hat{M}_- Y) = (b - \hbar)(\hat{M}_- Y)$$

$$\boxed{\begin{gathered}\hat{M}_z(\hat{M}_-^k Y) = (b - k\hbar)(\hat{M}_-^k Y) \\ k = 0, 1, 2, \ldots\end{gathered}}$$



$$\hat{M}^2 M_\pm^k Y = c M_\pm^k Y, \quad k = 0, 1, 2, \ldots$$

First show that $\hat{M}^2$ commutes with $\hat{M}_+$ and $\hat{M}_-$:

$$[\hat{M}^2, \hat{M}_\pm] = [\hat{M}^2, \hat{M}_x \pm i\hat{M}_y]$$

$$= [\hat{M}^2, \hat{M}_x] \pm i\,[\hat{M}^2, \hat{M}_y]$$

$$[\hat{M}^2, \hat{M}_\pm] = 0$$

$$[\hat{M}^2, \hat{M}_\pm{}^2] = \overbrace{[\hat{M}^2, \hat{M}_\pm]}^{0}\hat{M}_\pm + \hat{M}_\pm\overbrace{[\hat{M}^2, \hat{M}_\pm]}^{0}$$

$$[\hat{M}^2, \hat{M}_\pm{}^2] = 0$$

$$[\hat{M}^2, \hat{M}_\pm{}^k] = 0 \quad \text{or} \quad \boxed{\hat{M}^2\hat{M}_\pm^k = \hat{M}_\pm^k\hat{M}^2, \quad k = 0, 1, 2, \ldots}$$

$$\hat{M}^2 Y = cY$$

$$\Big\downarrow \ \text{apply } \hat{M}_\pm{}^k$$

$$\hat{M}_\pm{}^k \hat{M}^2 Y = \hat{M}_\pm{}^k c Y$$

$$\boxed{\hat{M}^2(\hat{M}_\pm{}^k Y) = c(\hat{M}_\pm{}^k Y)}$$

Show that the set of eigenvalues of $\hat{M}_z$ generated using the ladder operators must be bounded.

$$\hat{M}_z Y = bY$$

$$\hat{M}_z Y_k = b_k Y_k$$

where $Y_k = \hat{M}_\pm^k Y$

$$b_k = b \pm k\hbar$$

apply $\hat{M}_z$

$$\hat{M}_z^2 Y_k = b_k \hat{M}_z Y_k$$

$$\hat{M}_z^2 Y_k = b_k^2 Y_k$$

subtract from

$$\hat{M}^2 Y_k = cY_k$$

$$\hat{M}^2 Y_k - \hat{M}_z^2 Y_k = cY_k - b_k^2 Y_k$$

$$\underbrace{(\hat{M}_x^2 + \hat{M}_y^2)}_{} Y_k = (c - b_k^2) Y_k$$

nonnegative physical quantity with nonnegative eigenvalues

$$(c - b_k^2) \geq 0$$

$$c^{1/2} \geq |b_k|$$

$$c^{1/2} \geq b_k \geq -c^{1/2}, \quad k = 0, \pm 1, \pm 2, \ldots$$

remains constant as k varies

∴ The set of eigenvalues b_k is bounded above and below.

b_{max}, b_{min} : maximum and minimum values of b_k; Y_{max}, Y_{min} the corresponding eigenfunctions

$$\hat{M}_z Y_{max} = b_{max} Y_{max}$$

$$\hat{M}_z Y_{min} = b_{min} Y_{min}$$

$$\hat{M}_+ \hat{M}_z Y_{max} = b_{max} \hat{M}_+ Y_{max}$$

$$\hat{M}_z(\hat{M}_+ Y_{max}) = (b_{max} + \hbar)\underbrace{(\hat{M}_+ Y_{max})}_{\text{must go to zero}}$$

$$\hat{M}_+ Y_{max} = 0$$

operate with $\hat{M}_-$

$$\hat{M}_- \hat{M}_+ Y_{max} = 0$$

$$(\hat{M}_- \hat{M}_+ = \hat{M}^2 - \hat{M}_z^2 - \hbar \hat{M}_z)$$

$$(\hat{M}^2 - \hat{M}_z^2 - \hbar \hat{M}_z) Y_{max} = 0$$

$$(c - b_{max}^2 - \hbar b_{max}) Y_{max} = 0$$

$$c - b_{max}^2 - \hbar b_{max} = 0$$

$$\boxed{c = b_{max}^2 + \hbar b_{max}}$$

$$\hat{M}_- Y_{min} = 0$$

$$\boxed{c = b_{min}^2 - \hbar b_{min}}$$

$$b_{max}^2 + \hbar b_{max} + (\hbar b_{min} - b_{min}^2) = 0$$

quadratic in b_{max}

$$\boxed{b_{max} = -b_{min}}$$

$$\cancel{b_{max} = b_{min} - \hbar}$$

$$b_k = b \pm k\hbar$$

$$\boxed{b_{max} - b_{min} = n\hbar, \quad n = 0, 1, 2, \ldots}$$

$$b_{max} = \tfrac{1}{2} n\hbar$$

$$b_{max} = j\hbar, \quad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$b_{min} = -j\hbar$$

$$b = -j\hbar, (-j+1)\hbar, (-j+2)\hbar, \ldots, (j-2)\hbar,$$
$$(j-1)\hbar, j\hbar$$

$$c = b_{max}^2 + \hbar b_{max}$$

$$c = j(j+1)\hbar^2, \quad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

$$\boxed{\begin{aligned} &\hat{M}^2 Y = j(j+1)\hbar^2 Y, \quad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots \\ &\hat{M}_z Y = m_j \hbar Y, \quad m_j = -j, -j+1, \ldots, j-1, j \end{aligned}}$$

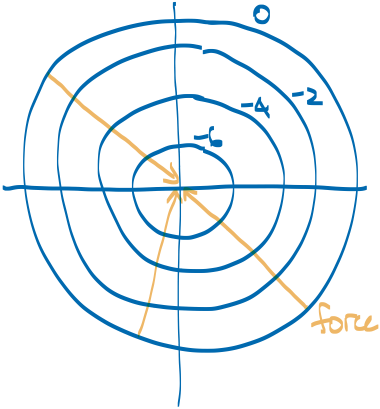
s	p	d
$l = 0$	$l = 1$	$l = 2$
$m_l = 0$	$m_l = -1, 0, 1$	$m_l = -2, -1, 0, 1, 2$

Chapter 6: The Hydrogen Atom

6.1 The One-Particle Central-Force Problem

- **central force**: derived from a potential energy function that is sphericaly symmetric

$$V = V(r)$$



$$\vec{F} = -\nabla V(x,y,z)$$

$$= -\left(\frac{\partial V}{\partial x}\right)\vec{i} - \left(\frac{\partial V}{\partial y}\right)\vec{j} - \left(\frac{\partial V}{\partial z}\right)\vec{k}$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)_{y,z}\frac{\partial}{\partial r} + \cancel{\left(\frac{\partial \theta}{\partial x}\right)_{y,z}\frac{\partial}{\partial \theta}} + \cancel{\left(\frac{\partial \phi}{\partial x}\right)_{y,z}\frac{\partial}{\partial \phi}}$$

function only depends on r

$\left(\frac{\partial V}{\partial \theta}\right)_{r,\phi} = 0$ and $\left(\frac{\partial V}{\partial \phi}\right)_{r,\theta} = 0$

$$\left(\frac{\partial V}{\partial x}\right)_{y,z} = \frac{dV}{dr}\boxed{\left(\frac{\partial r}{\partial x}\right)_{y,z}} \rightarrow \frac{x}{r}$$

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \sin\theta\cos\phi$$

$$x = r\sin\theta\cos\phi$$

$$\boxed{\left(\frac{\partial V}{\partial x}\right)_{y,z} = \frac{x}{r}\frac{dV}{dr}}$$

$$\boxed{\left(\frac{\partial V}{\partial y}\right)_{x,z} = \frac{y}{r}\frac{dV}{dr}, \quad \left(\frac{\partial V}{\partial z}\right)_{x,y} = \frac{z}{r}\frac{dV}{dr}}$$

$$\vec{F} = -\frac{1}{r}\frac{dV}{dr}(x\vec{i}+y\vec{j}+z\vec{k})$$

$$\boxed{\vec{F} = -\frac{dV(r)}{dr}\frac{\vec{r}}{r}}$$

$\frac{\vec{r}}{r}$: unit vector in the radial direction

$$\hat{H} = \hat{T}+\hat{V} = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

Need the Laplacian in spherical coordinates

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\underbrace{\frac{\partial^2}{\partial \theta^2}}_{} + \frac{1}{r^2}\underbrace{\cot\theta\frac{\partial}{\partial \theta}}_{} + \frac{1}{r^2 \underbrace{\sin^2\theta}_{}}\underbrace{\frac{\partial^2}{\partial \phi^2}}_{}}$$

$$\hat{L}^2 = -\hbar^2\left(\underbrace{\frac{\partial^2}{\partial \theta^2}}_{} + \underbrace{\cot\theta\frac{\partial}{\partial \theta}}_{} + \underbrace{\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}}_{}\right)$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2\hbar^2}\hat{L}^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{1}{2mr^2}\hat{L}^2 + V(r)$$

In classical mechanics, a particle subject to a central force has its angular momentum conserved. In quantum mechanics, can we have definite values for the energy and angular momentum?

$$[\hat{H}, \hat{L}^2] = ?$$