Lecture 19

Wednesday, November 29, 2023 15:58

Practice Problems

1. Let $\vec{A}$ have the components $(3,-2,6)$; let $\vec{B}$ have the components $(-1,4,4)$. Find $|\vec{A}|$, $|\vec{B}|$, $\vec{A}+\vec{B}$, $\vec{A}-\vec{B}$, $\vec{A}\cdot\vec{B}$, $\vec{A}\times\vec{B}$. Find the angle between $\vec{A}$ and $\vec{B}$.

$$|\vec{A}| = \left[3^2+(-2)^2+6^2\right]^{\frac{1}{2}} = 7$$

$$|\vec{B}| = \left[(-1)^2+4^2+4^2\right]^{\frac{1}{2}} = \sqrt{33}$$

$$\vec{A}+\vec{B} = [3+(-1)]\vec{i}+[(-2)+4]\vec{j}+(6+4)\vec{k}$$

$$= 2\vec{i}+2\vec{j}+10\vec{k} \quad \text{or} \quad (2,2,10)$$

$$\vec{A}-\vec{B} = [3-(-1)]\vec{i}+[(-2)-4]\vec{j}+(6-4)\vec{k}$$

$$= 4\vec{i}-6\vec{j}+2\vec{k} \quad \text{or} \quad (4,-6,2)$$

$$\vec{A} \cdot \vec{B} = (3)(-1) + (-2)(4) + (6)(4) = 13$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 6 \\ -1 & 4 & 4 \end{vmatrix}$$

$$= (A_2B_3 - A_3B_2)\vec{i} - (A_1B_3 - A_3B_1)\vec{j}$$

$$+ (A_1B_2 - A_2B_1)\vec{k}$$

$$= [(-2)(4) - (6)(4)]\vec{i} - [(3)(4) - (6)(-1)]\vec{j}$$

$$+ [(3)(4) - (-2)(-1)]\vec{k}$$

$$= -32\vec{i} - 18\vec{j} + 10\vec{k} \text{ or } (-32, -18, 10)$$

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

$$13 = 7\sqrt{33} \cos\theta$$

$\theta = \arccos(0.323) = 1.24$ rad or $71°$

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2. Find $[\hat{L}_x^2, \hat{L}_y]$.

$$[\hat{L}_x^2, \hat{L}_y] = \hat{L}_x \underbrace{[\hat{L}_x, \hat{L}_y]}_{i\hbar\hat{L}_z} + [\hat{L}_x, \hat{L}_y]\hat{L}_x$$

$$= \hat{L}_x i\hbar \hat{L}_z + i\hbar \hat{L}_z L_x$$

$$= \boxed{i\hbar(\hat{L}_x\hat{L}_z + \hat{L}_z\hat{L}_x)}$$

$$[\hat{L}_z, \hat{L}_x] = \hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z$$

$$\hat{L}_z\hat{L}_x = \boxed{\underbrace{[\hat{L}_z, \hat{L}_x]}_{i\hbar\hat{L}_y} + \hat{L}_x\hat{L}_z}$$

$$\boxed{[\hat{L}_x^2, \hat{L}_y] = i\hbar(i\hbar L_y + 2\hat{L}_x\hat{L}_z)}$$

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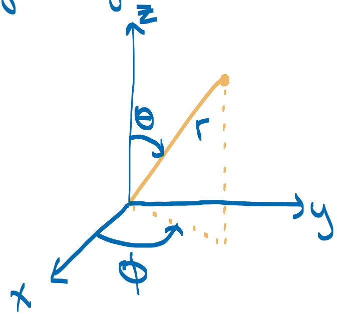
3. Find the spherical coordinates for points with the following (x,y,z) coordinates.



$$r = (x^2+y^2+z^2)^{\frac{1}{2}}, \quad \cos\theta = \frac{z}{r}, \quad \tan\phi = \frac{y}{x}$$

a) $(1,2,0)$

$$\boxed{r = [1^2+2^2]^{\frac{1}{2}} = \sqrt{5}}$$

$$\cos\theta = \frac{0}{\sqrt{5}} = 0, \quad \boxed{\theta = \frac{\pi}{2} = 90^\circ}$$

$$\tan\phi = \frac{2}{1} = 2, \quad \boxed{\phi = \tan^{-1} 2 \approx 63^\circ}$$

b) $(-1,0,3)$

$$\boxed{r = \left[(-1)^2 + 3^2\right]^{\frac{1}{2}} = \sqrt{10}}$$

$$\cos\theta = \frac{3}{\sqrt{10}} \qquad \boxed{\theta = \cos^{-1}\frac{3}{\sqrt{10}} = 19^\circ}$$

$$\tan\phi = 0 \qquad \boxed{\phi = \tan^{-1} 0 = 0^\circ}$$

c) $(3, 1, -2)$

$$\boxed{r = \left[3^2 + 1^2 + (-2)^2\right]^{\frac{1}{2}} = \sqrt{14}}$$

$$\cos\theta = \frac{z}{r} = -\frac{2}{\sqrt{14}} \qquad \boxed{\theta = \cos^{-1}\left(-\frac{2}{\sqrt{14}}\right) = 122^\circ}$$

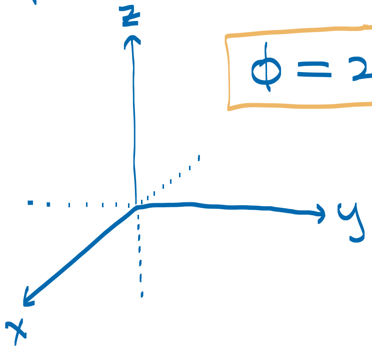
$$\tan\phi = \frac{1}{3} \qquad \boxed{\phi = \tan^{-1}\frac{1}{3} = 15^\circ}$$

d) $(-1, -1, -1)$

$$\boxed{r = \left[3(-1)^2\right]^{\frac{1}{2}} = \sqrt{3}}$$

$$\cos\theta = \frac{z}{r} = -\frac{1}{\sqrt{3}} \qquad \boxed{\theta = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right) = 125^\circ}$$

$\tan\phi = 1 \quad \phi = \tan^{-1} 1 = 45^\circ$



$\boxed{\phi = 225^\circ}$

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4. Find the (x, y, z) coordinates of the points with the following spherical coordinates.

a) $r = 1, \theta = \frac{\pi}{2}, \phi = \pi$

$x = r\sin\theta\cos\phi$

$= (1)\sin\frac{\pi}{2}\cos\pi$

$= -1$

$y = r\sin\theta\sin\phi$

$= (1)\sin\frac{\pi}{2}\sin\pi$

$= 0$

$z = r\cos\theta$

$= (1)\cos\frac{\pi}{2}$

$= 0$

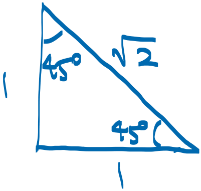
$$\boxed{\therefore (-1, 0, 0)}$$

b) $r = 2,\ \theta = \frac{\pi}{4},\ \phi = 0$

$x = r\sin\theta\cos\phi$

$= (2)\sin\frac{\pi}{4}\cos 0$

$= \frac{2}{\sqrt{2}} = \frac{2^1}{2^{\frac{1}{2}}} = \sqrt{2}$



$y = r\sin\theta\sin\phi$

$= (2)\sin\frac{\pi}{4}\sin 0$

$= 0$

$z = r\cos\theta$

$= (2)\cos\frac{\pi}{4}$

$= \sqrt{2}$

$$\boxed{\therefore (\sqrt{2}, 0, \sqrt{2})}$$

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5. Calculate the possible angles between $\vec{L}$ and the z axis for $l = 2$.

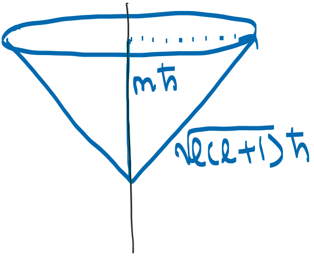
$l = 2, \quad m = -2, -1, 0, 1, 2$

$\uparrow \qquad\qquad \uparrow$

$\hat{L}^2 \qquad\qquad \hat{L}_z$

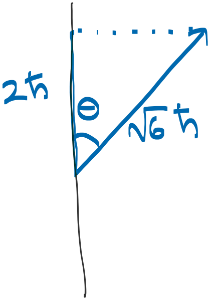
eigenvalues of $\hat{L}^2$: $l(l+1)\hbar^2$

eigenvalues of $\hat{L}_z$: $m\hbar$



$\sqrt{l(l+1)}\,\hbar = \sqrt{6}\,\hbar$

$m=2$

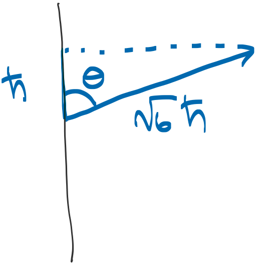


$$\cos\theta = \frac{2}{\sqrt{6}}$$

$$\theta = \cos^{-1}\frac{2}{\sqrt{6}}$$

$$\boxed{\theta = 35^\circ}$$

$m=1$

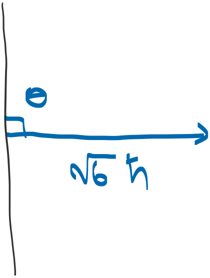


$$\cos\theta = \frac{1}{\sqrt{6}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{6}}$$

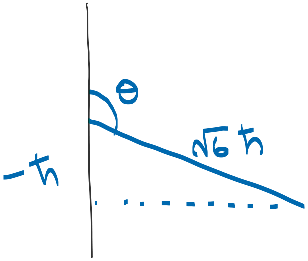
$$\boxed{\theta = 66^\circ}$$

$m=0$



$$\boxed{\theta = 90^\circ}$$

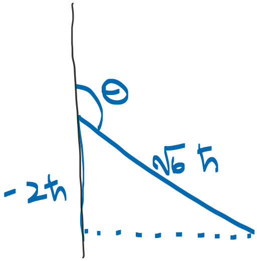
$m=-1$



$$\theta = 180° - 66°$$

$$\boxed{\theta = 114°}$$

$m = -2$



$$\theta = 180° - 35°$$

$$\boxed{\theta = 145°}$$

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6. Show that the spherical harmonics are eigenfunctions of the operator $\hat{L}_x^2 + \hat{L}_y^2$. What are the eigenvalues?

$$\hat{L}_x^2 + \hat{L}_y^2 = \hat{L}^2 - \hat{L}_z^2$$

$$(\hat{L}^2 - \hat{L}_z^2) Y_\ell^m = \hat{L}^2 Y_\ell^m - \hat{L}_z \hat{L}_z Y_\ell^m$$

$$= \ell(\ell+1)\hbar^2 Y_\ell^m - m^2\hbar^2 Y_\ell^m$$

$$\boxed{\therefore (\hat{L}_x^2 + \hat{L}_y^2) Y_\ell^m = [\ell(\ell+1)\hbar - m^2\hbar^2] Y_\ell^m}$$

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7. a) If we measure Lz of a particle that has angular momentum quantum number $\ell = 2$, what are the possible outcomes of the measurement?

$$\ell = 2, \quad m = -2, -1, 0, 1, 2$$

$$\hat{L}_z Y_\ell^m = m\hbar Y_\ell^m$$

$$\boxed{\therefore -2\hbar, -\hbar, 0, \hbar, 2\hbar}$$

b) If we measure Lz of a particle whose state function is an eigenfunction of $\hat{L}^2$ with eigenvalue $12\hbar^2$, what are the possible outcomes of the measurement?

$$\ell(\ell+1)\hbar^2 = 12\hbar^2 \; ; \quad \ell = 3$$

$$\boxed{\therefore -3\hbar, -2\hbar, -\hbar, 0, \hbar, 2\hbar, 3\hbar}$$

8. Apply the lowering operator $\hat{L}_-$ three times in succession to $Y_1^1(\theta,\phi)$ and verify that we obtain functions that are proportional to Y_1^0, Y_1^{-1}, and zero.

$$Y_1^1 = \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta\, e^{i\phi}$$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta$$

$$Y_1^{-1} = \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta\, e^{-i\phi}$$

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y$$

$$= \hbar\left[(i\sin\phi - \cos\phi)\frac{\partial}{\partial\theta} + \cot\theta\, i(\cos\phi - i\sin\phi)\frac{\partial}{\partial\phi}\right]$$

$$\boxed{\hat{L}_- = \hbar\left(-e^{-i\phi}\frac{\partial}{\partial\theta} + i\cot\theta\, e^{-i\phi}\frac{\partial}{\partial\phi}\right)}$$

$$\hat{L}_- Y_1^1 = \hbar\left(-e^{-i\phi}\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}e^{-i\phi}\frac{\partial}{\partial\phi}\right)\frac{1}{2}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}} e^{i\phi}\sin\theta$$

$$= \hbar\left(-\cancel{e^{-i\phi}\, e^{i\phi}}\,\frac{1}{2}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\frac{\partial}{\partial\theta}\overset{\swarrow \cos\theta}{\sin\theta}\right.$$

$$\left. + i\frac{\cos\theta}{\cancel{\sin\theta}}\, e^{-i\phi}\frac{\partial}{\partial\phi}\,\frac{1}{2}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}} e^{i\phi}\,\cancel{\sin\theta}\right)$$

$$= \hbar\left(-\frac{1}{2}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\cos\theta + i\overbrace{\cos\theta\, e^{-i\phi}\, i}^{-1}\,\underbrace{\frac{1}{2}\left(\frac{3}{2\pi}\right)^{\frac{1}{2}} e^{i\phi}}_{1}\right)$$

$$\boxed{\hat{L}_- Y_1^1 = -\hbar\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\cos\theta} \propto Y_1^0$$

$$Y_1^0\rangle$$

$$\hat{L}_- Y_1^0\rangle = \hbar\left(-e^{-i\phi}\frac{\partial}{\partial\theta} + i\cot\theta\, e^{-i\phi}\frac{\partial}{\partial\phi}\right)$$

$$\left[-\hbar\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\cos\theta\right]$$

$$= \hbar\left[\cancel{-}\,e^{-i\phi}\frac{\partial}{\partial\theta}\,\cancel{-}\hbar\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\cos\theta\right.$$

$$\left. + i\frac{\cos\theta}{\sin\theta}\, e^{-i\phi}\frac{\partial}{\partial\phi} - \hbar\left(\frac{3}{2\pi}\right)^{\frac{1}{2}}\cos\theta\right]$$

$$\boxed{\hat{L}_- Y_1^0\rangle = -\hbar^2\left(\frac{3}{2\pi}\right)^{\frac{1}{2}} e^{-i\phi}\sin\theta} \propto Y_1^{-1}$$

(The last application of $\hat{L}_-$ is left as an exercise to the reader.)