Lecture 9

Wednesday, October 11, 2023 15:51

Chapter 4 : The Harmonic Oscillator continued

4.1 Power-Series Solution of Differential

Equations

$$\boxed{y''(x) + c^2 y(x) = 0} \qquad c^2 > 0$$

constant coefficients

auxiliary equation $r^2 + c^2 = 0$

$$r = \pm ic$$

$$\boxed{y = A\cos cx + B\sin cx}$$

Now, let's solve using the power-series method.

Apply Taylor expansion

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \qquad \text{about } x=0$$

(assuming that $y(x)$ can be expanded in this way)

$$\boxed{y(x)} = \boxed{\sum_{n=0}^{\infty} a_n x^n} = a_0 + a_1 x + a_2 x^2 + \cdots$$

where the a's are constant coefficients

differentiate term-by-term

$$y'(x) = a_1 + 2a_2 x + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\boxed{y''(x)} = 2a_2 + 3(2)a_3 x + \cdots = \boxed{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} c^2 a_n x^n = 0$$

combine term-by-term

$$\sum_{j=0}^{\infty} b_j x^j + \sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} (b_j + c_j) x^j$$

change summation index in first sum

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k$$

$$k \equiv n-2$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\,x^n = y''(x)$$

$$\sum_{n=0}^{\infty} \underbrace{\left[(n+2)(n+1)a_{n+2} + c^2 a_n\right]} x^n = 0$$

for any value of x, if this equation holds true, the coefficients must all vanish

$$\therefore (n+2)(n+1)a_{n+2} + c^2 a_n = 0$$

$$\boxed{a_{n+2} = -\frac{c^2}{(n+1)(n+2)}\,a_n}$$

a recursion relation

no restrictions on a_0 and a_1; thus, arbitrary constants

$$a_0 = A, \qquad a_1 = Bc$$

$$a_0 = A, \; a_2 = -\frac{c^2 A}{1 \cdot 2}, \; a_4 = \frac{c^4 A}{4 \cdot 3 \cdot 2 \cdot 1},$$

$$a_6 = -\frac{c^6 A}{6!}, \; \ldots$$

$$\boxed{a_{2k} = (-1)^k \frac{c^{2k} A}{(2k)!}, \quad k = 0, 1, 2, 3, \ldots}$$

$$a_1 = Bc, \; a_3 = -\frac{c^3 B}{2 \cdot 3}, \; a_5 = \frac{c^5 B}{5 \cdot 4 \cdot 3 \cdot 2},$$

$$a_7 = -\frac{c^7 B}{7!}, \; \ldots$$

$$\boxed{a_{2k+1} = (-1)^k \frac{c^{2k+1} B}{(2k+1)!}, \quad k = 0, 1, 2, 3, \ldots}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0,2,4}^{\infty} a_n x^n + \sum_{n=1,3,5}^{\infty} a_n x^n$$

$$\boxed{y = \underbrace{A\sum_{k=0}^{\infty}(-1)^k \frac{c^{2k}x^{2k}}{(2k)!}}_{\text{cosine}} + \underbrace{B\sum_{k=0}^{\infty}(-1)^k \frac{c^{2k+1}x^{2k+1}}{(2k+1)!}}_{\text{sine}}}$$

$$\boxed{y = A\cos cx + B\sin cx}$$

4.2 The One-Dimensional Harmonic Oscillator

Quantum Mechanical Treatment

$$\hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + 2\pi^2\nu^2 m x^2$$

previously: $V = 2\pi^2\nu^2 m x^2$

$$\alpha \equiv \frac{2\pi\nu m}{\hbar}$$

multiply by $\frac{2m}{\hbar^2}$

$$\hat{H} = -\frac{d^2}{dx^2} + \overset{\alpha^2}{\boxed{\frac{4\pi^2\nu^2 m^2}{\hbar^2}}}\,x^2$$

$$\boxed{\frac{d^2\psi}{dx^2} + \left(2mE\hbar^2 - \alpha^2 x^2\right)\psi = 0}$$

need to rewrite so we get a two-term recursion relation

$$f(x) \equiv e^{\alpha x^2/2}\,\psi(x)$$

$$\boxed{\psi = e^{-\alpha x^2/2}\, f(x)}$$

$$\psi' = e^{-\alpha x^2/2}\, f'(x) + \left(-\frac{\alpha}{2}\cdot 2x\right) e^{-\alpha x^2/2}\, f(x)$$

$$\boxed{\psi' = e^{-\alpha x^2/2} f'(x) - \alpha x\, e^{-\alpha x^2/2}\, f(x)}$$

$$\psi'' = e^{-\alpha x^2/2}\, f''(x) - \alpha x\, e^{-\alpha x^2/2}\, f'(x)$$

$$- \left[\alpha\, e^{-\alpha x^2/2}\, f(x) + \alpha x\, e^{-\alpha x^2/2}\, f'(x)\right.$$

$$\left. - \alpha^2 x^2 e^{-\alpha x^2/2}\, f(x)\right]$$

$$\psi'' = e^{-\alpha x^2/2} f''(x) - 2\alpha x e^{-\alpha x^2/2} f'(x)$$
$$- \alpha e^{-\alpha x^2/2} f(x) + \alpha^2 x^2 e^{-\alpha x^2/2} f(x)$$

$$\boxed{\psi'' = e^{-\alpha x^2/2}\left(f'' - 2\alpha x f' - \alpha f + \alpha^2 x^2 f\right)}$$

$$\cancel{e^{-\alpha x^2/2}}\left(f'' - 2\alpha x f' - \alpha f + \alpha^2 x^2 f\right)$$
$$+ \left(2mE\hbar^{-2} - \alpha^2 x^2\right) \cancel{e^{-\alpha x^2/2}} f(x) = 0$$

$$f'' - 2\alpha x f' - \alpha f + \cancel{\alpha^2 x^2} f$$
$$+ 2mE\hbar^{-2} f - \cancel{\alpha^2 x^2} f = 0$$

$$\boxed{f'' - 2\alpha x f' + \left(2mE\hbar^{-2} - \alpha\right) f = 0}$$

power-series solution:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$f''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n - 2\alpha \sum_{n=0}^{\infty} n\, c_n x^n$$

$$+ \left(2mE\hbar^{-2} - \alpha\right) \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \Big[\underline{(n+2)(n+1)\, c_{n+2} - 2\alpha n\, c_n}$$

$$\underline{+ \left(2mE\hbar^{-2} - \alpha\right) c_n} \Big] x^n = 0$$

$$(n+2)(n+1)\, c_{n+2} - 2\alpha n\, c_n$$

$$+ \left(2mE\hbar^{-2} - \alpha\right) c_n = 0$$

$$\boxed{c_{n+2} = \frac{\alpha + 2\alpha n - 2mE\hbar^{-2}}{(n+2)(n+1)}\, c_n}$$

set $c_1 = 0$

$$\psi_1 = e^{-\alpha x^2/2} f(x)$$

$$= e^{-\alpha x^2/2} \sum_{n=0,2,4,\ldots}^{\infty} c_n x^n = e^{-\alpha x^2/2} \sum_{l=0}^{\infty} c_{2l} x^{2l}$$

set $c_0 = 0$:

$$\psi_2 = e^{-\alpha x^2/2} \sum_{n=1,3,5,\ldots}^{\infty} c_n x^n = e^{-\alpha x^2/2} \sum_{l=0}^{\infty} c_{2l+1} x^{2l+1}$$

$$\boxed{\psi = A e^{-\alpha x^2/2} \sum_{l=0}^{\infty} c_{2l+1} x^{2l+1} + B e^{-\alpha x^2/2} \sum_{l=0}^{\infty} c_{2l} x^{2l}}$$

how does the series behave for large x?

$$c_{n+2} = \frac{\alpha + 2\alpha n - 2mE\hbar^{-2}}{(n+1)(n+2)} c_n$$

$n = 2l$

$$\frac{c_{2l+2}}{c_{2l}} = \frac{\cancel{\alpha} + 4\alpha l - \cancel{2mE\hbar^{-2}}}{(2l+\cancel{1})(2l\cancel{+2})}$$

in the limit of large ℓ:

$$\boxed{\frac{C_{2\ell+2}}{C_{2\ell}} \approx \frac{4\alpha\ell}{(2\ell)(2\ell)} = \frac{\alpha}{\ell}}$$

for ℓ large

$$e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

$$e^{\alpha x^2} = 1 + \alpha x^2 + \cdots + \underbrace{\frac{\alpha^{\ell} x^{2\ell}}{\ell!}} + \underbrace{\frac{\alpha^{\ell+1} x^{2\ell+2}}{(\ell+1)!}}$$

$$+ \cdots$$

$$\frac{\alpha^{\ell+1}}{(\ell+1)!} \Big/ \frac{\alpha^{\ell}}{\ell!} = \frac{x^{2\ell+2}}{x^{2\ell}}$$

$$\frac{\alpha}{\ell+1} \approx \boxed{\frac{\alpha}{\ell}}$$

Therefore, for large x, each series behaves as $e^{\alpha x^2}$.

$$\Psi = A e^{-\alpha x^2/2} \cdot e^{\alpha x^2} + B e^{-\alpha x^2/2} e^{\alpha x^2}$$

$$\psi = A e^{\alpha x^2/2} + B e^{\alpha x^2/2}$$

break off the series after a finite number of terms; i.e. c_n must vanish for some value of n, say $n = v$.

set to 0

$$c_{n+2} = \frac{\boxed{\alpha + 2\alpha n - 2mE\hbar^{-2}}}{(n+2)(n+1)} c_n$$

$$\alpha + 2\alpha v - 2mE\hbar^{-2} = 0$$

$$2mE\hbar^{-2} = 2\alpha v + \alpha$$

$$\left(\alpha \equiv \frac{2\pi\nu m}{\hbar}\right)$$

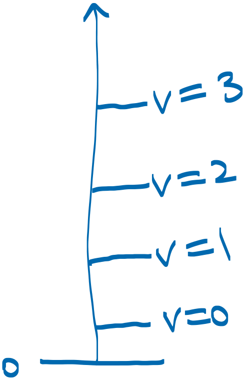
$$2mE\hbar^{-2} = \frac{4\pi\nu m}{\hbar} v + \frac{2\pi\nu m}{\hbar}$$

$$\cancel{2}\cancel{m} E \hbar^{-\overset{1}{\cancel{2}}} = (2v+1)\cancel{2}\pi\nu \cancel{m} \cancel{\hbar^{-1}}$$

$$E = (2v+1)\pi\nu\hbar$$

$$E = \frac{(2v+1)\cancel{\pi}\nu h}{2\cancel{\pi}}$$

$$\boxed{E = (v+\frac{1}{2})h\nu}$$



energy levels of the harmonic oscillator